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时间尺度联合分数阶约束力学系统 Noether 定理*

宋传静[1] 吴镇宇[1] 张毅[2†]
(1. 苏州科技大学 数学科学学院, 苏州 215009)
(2. 苏州科技大学 土木工程学院, 苏州 215011)

摘要 研究时间尺度联合 Caputo 导数下的 Lagrange 系统和 Hamilton 系统的 Noether 定理. 首先,给出时间尺度联合 Caputo 导数的定义;其次,研究时间尺度联合 Caputo 导数下的系统运动微分方程,包括 Lagrange 系统和 Hamilton 系统;第三,分别给出这两个系统的 Noether 恒等式和守恒量;第四,对于所得结果及所使用的方法,各自举例进行说明.

关键词 时间尺度, 联合 Caputo 导数, Noether 定理, Lagrange 系统, Hamilton 系统
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Noether's Theorem of Combined Fractional Constrained Mechanical Systems on Time Scales*

Song Chuanjing[1] Wu Zhenyu[1] Zhang Yi[2†]
(1. School of Mathematical Sciences, Suzhou University of Science and Technology, Suzhou 215009, China)
(2. College of Civil Engineering, Suzhou University of Science and Technology, Suzhou 215011, China)

Abstract Noether theorems for Lagrangian and Hamiltonian systems in terms of combined Caputo derivative on time scales are investigated. Firstly, the definition of the combined Caputo derivative on time scales is given; Secondly, differential equations of motion in terms of the combined Caputo derivative on time scales are presented; Thirdly, Noether identities and conserved quantities for those two systems are studied separately; Fourthly, examples are provided.

Key words time scale, combined Caputo derivative, Noether theorem, Lagrangian system, Hamiltonian system

引言

1918 年 Emmy Noether 发表论文《不变变分问题》,证明了 Noether 定理[1]. Noether 定理不仅在理论物理和数学分析中具有重要地位,还在量子力学和广义相对论中有广泛应用,极大地推动了科学的发展. Noether 定理的研究在 Lagrange 系统、Hamilton 系统和 Birkhoff 系统中取得了一系列成果[2-9].

时间尺度的概念于 1988 年被首次提出[10-12].

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† 通信作者 E-mail:zhy@mail.usts.edu.cn

实数集、整数集、自然数集、闭区间的并集及康托尔集等均是时间尺度的例子.由于实数集和整数集可以同时被概括在时间尺度里,因此说时间尺度具有统一的特征.另一方面,时间尺度上的研究成果,可以根据需要,选择特殊的定义域进行具体化,定义域不同,所得结果也有各自的特点,因此说时间尺度具有拓展的特征.时间尺度 Noether 定理的研究始于 Bartosiewicz[13].时间尺度上 Noether 定理的研究涉及 Lagrange 系统[14-16]、Hamilton 系统[17-19]、Birkhoff 系统[20-22]及非完整系统[23-24]等.

分数阶微积分的提出早在 1695 年,它与整数阶微积分的提出在同一个时代.分数阶导数的类型多种多样.2015 年,Luo 和 Xu[25]整理了比较广泛的联合分数阶导数.分数阶 Noether 定理的研究始于 Frederico 和 Torres[26],且取得一系列研究成果[27-33].

时间尺度分数阶方面的研究也取得一些成果,如时间尺度分数阶原理和不等式[34]、时间尺度分数阶动力方程[35]、时间尺度分数阶混沌系统[36]、时间尺度分数阶递归神经网络[37]等.时间尺度分数阶约束力学系统对称性的研究刚刚起步,目前仅研究了时间尺度 Caputo 分数阶约束力学系统的 Noether 定理[38-40].考虑到联合分数阶导数的普适性和广泛性,本文拟研究时间尺度联合分数阶约束力学系统的 Noether 定理.

文章结构安排如下:第一节列出时间尺度分数阶微积分必要的预备知识;第二节根据变分原理建立时间尺度联合 Caputo 分数阶 Lagrange 方程和时间尺度联合 Caputo 分数阶 Hamilton 方程;第三节研究 Noether 恒等式,并进一步给出时间尺度联合 Caputo 分数阶 Lagrange 系统和时间尺度联合 Caputo 分数阶 Hamilton 系统的 Noether 定理;第四节给出例题,通过举例说明本文所得结果及所使用的方法.

1 预备知识

定义 1[41] 设 $\mathbb{T}$为时间尺度,函数 $f:[a,b]\to\mathbb{R}$,$\alpha\geqslant 0$,则基于时间尺度的左、右 Riemann-Liouville 积分${}_aI^{\alpha}_{\Delta,t}f(t)$,${}_tI^{\alpha}_{\Delta,b}f(t)$,分别为

$\alpha=0$,${}_aI^{\alpha}_{\Delta,t}f(t)=f(t)$;$\alpha>0$,

$$ {}_aI^{\alpha}_{\Delta,t}f(t)=\int_a^t h_{\alpha-1}[t,\sigma(\tau)]f(\tau)\Delta\tau \tag{1}$$

和

$\alpha=0$,${}_tI^{\alpha}_{\Delta,b}f(t)=f(t)$;$\alpha>0$,

$$ {}_tI^{\alpha}_{\Delta,b}f(t)=\int_t^b h_{\alpha-1}[\sigma(\tau),t]f(\tau)\Delta\tau \tag{2}$$

其中,$h_\alpha:\mathbb{T}\times\mathbb{T}\to\mathbb{R}$,$\alpha\geqslant 0$ 为右连续函数,满足

$$ h_{\alpha+1}(t,s)=\int_s^t h_\alpha(\tau,s)\Delta\tau,\ h_0(t,s)=1,\quad t,s\in\mathbb{T} \tag{3}$$

下文将 Riemann-Liouville 简写为 R-L.

注 1 当 $\mathbb{T}=\mathbb{R}$ 时,有 $h_{\alpha-1}(t,a)=(t-a)^{\alpha-1}/\Gamma(\alpha)$,$h_{\alpha-1}(b,t)=(b-t)^{\alpha-1}/\Gamma(\alpha)$ 时,此时有

$$ {}_aI^{\alpha}_tf(t)=\int_a^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau)\mathrm{d}\tau,\ t>a;$$

$$ {}_tI^{\alpha}_bf(t)=\int_t^b\frac{(\tau-t)^{\alpha-1}}{\Gamma(\alpha)}f(\tau)\mathrm{d}\tau,\ t<b \tag{4}$$

式(4)即为传统的左、右 R-L 积分.

定义 2[41] 设 $\mathbb{T}$是时间尺度,函数 $f:\mathbb{T}\to\mathbb{R}$,$m-1\leqslant\alpha<m$,$m\in\mathbb{N}_+$,时间尺度左、右 R-L 导数和时间尺度左、右 Caputo 导数分别为

$$ {}^{\mathrm{RL}}_{\ a}D^{\alpha}_{\Delta,t}f(t)=D^m_\Delta[{}_aI^{m-\alpha}_{\Delta,t}f(t)]=D^m_\Delta\{\int_a^t h_{m-\alpha-1}[t,\sigma(\tau)]f(\tau)\Delta\tau\} \tag{5}$$

$$ {}^{\mathrm{RL}}_{\ t}D^{\alpha}_{\Delta,b}f(t)=(-1)^mD^m_\Delta[{}_tI^{m-\alpha}_{\Delta,b}f(t)]=(-1)^mD^m_\Delta\{\int_t^b h_{m-\alpha-1}[\sigma(\tau),t]f(\tau)\Delta\tau\} \tag{6}$$

$$ {}^{\mathrm{C}}_{a}D^{\alpha}_{\Delta,t}f(t)={}_aI^{m-\alpha}_{\Delta,t}[D^m_\Delta f(t)]=\int_a^t h_{m-\alpha-1}[t,\sigma(\tau)][D^m_\Delta f(\tau)]\Delta\tau \tag{7}$$

$$ {}^{\mathrm{C}}_{t}D^{\alpha}_{\Delta,b}f(t)=(-1)^m{}_tI^{m-\alpha}_{\Delta,b}[D^m_\Delta f(t)]=(-1)^m\int_t^b h_{m-\alpha-1}[\sigma(\tau),t][D^m_\Delta f(\tau)]\Delta\tau \tag{8}$$

其中 D^m_Δ 表示时间尺度上求 m 阶导数.

注 2 当 $\mathbb{T}=\mathbb{R}$ 时,式(5)~(8)分别退化为传统的左、右 R-L 导数和左、右 Caputo 导数.

时间尺度联合 R-L 导数和时间尺度联合 Caputo 导数可定义为

$$ {}^{\mathrm{RL}}D^{\alpha,\beta}_{\Delta,\gamma}f(t)=\gamma\,{}^{\mathrm{RL}}_{\ a}D^{\alpha}_{\Delta,t}f(t)+(-1)^n(1-\gamma)\,{}^{\mathrm{RL}}_{\ t}D^{\beta}_{\Delta,b}f(t) \tag{9}$$

$$ {}^{\mathrm{C}}D^{\alpha,\beta}_{\Delta,\gamma}f(t)=\gamma\,{}^{\mathrm{C}}_{a}D^{\alpha}_{\Delta,t}f(t)+(-1)^n(1-\gamma)\,{}^{\mathrm{C}}_{t}D^{\beta}_{\Delta,b}f(t) \tag{10}$$

其中,$n-1\leqslant\alpha,\beta<n$,$\gamma\in[0,1]$.

特别地,本文中均令 $n=1$,则有 $0\leqslant\alpha,\beta<1$.

注 3 当 $\gamma=1$ 时,由式(9)和式(10)可得

$$ {}^{\mathrm{RL}}D^{\alpha,\beta}_{\Delta,\gamma}f(t)={}^{\mathrm{RL}}_{\ a}D^{\alpha}_{\Delta,t}f(t),\ {}^{\mathrm{C}}D^{\alpha,\beta}_{\Delta,\gamma}f(t)={}^{\mathrm{C}}_{a}D^{\alpha}_{\Delta,t}f(t) \tag{11}$$

即时间尺度联合 R-L 导数和时间尺度联合 Caputo 导数分别退化为时间尺度左 R-L 和时间尺度左 Caputo 导数. 当 $\gamma=0$ 时,由式(9)和式(10)可得

$$
{}^{\mathrm{RL}}D^{\alpha,\beta}_{\Delta,\gamma}f(t)=-{}^{\mathrm{RL}}_{t}D^{\beta}_{\Delta,b}f(t),
$$
$$
{}^{\mathrm{C}}D^{\alpha,\beta}_{\Delta,\gamma}f(t)=-{}^{\mathrm{C}}_{t}D^{\beta}_{\Delta,b}f(t) \tag{12}
$$

即时间尺度联合 R-L 导数和时间尺度联合 Caputo 导数分别退化为时间尺度右 R-L 和右 Caputo 导数. 当 $\alpha=\beta,\gamma=1/2$ 时,由式(9)和式(10)可得

$$
{}^{\mathrm{RL}}D^{\alpha,\beta}_{\Delta,1/2}f(t)=\frac{1}{2}{}^{\mathrm{RL}}_{a}D^{\alpha}_{\Delta,t}f(t)-\frac{1}{2}{}^{\mathrm{RL}}_{t}D^{\beta}_{\Delta,b}f(t)={}^{\mathrm{RL}}_{a}D^{\alpha}_{\Delta,b}f(t) \tag{13}
$$
$$
{}^{\mathrm{C}}D^{\alpha,\beta}_{\Delta,1/2}f(t)=\frac{1}{2}{}^{\mathrm{C}}_{a}D^{\alpha}_{\Delta,t}f(t)-\frac{1}{2}{}^{\mathrm{C}}_{t}D^{\beta}_{\Delta,b}f(t)={}^{\mathrm{C}}_{a}D^{\alpha}_{\Delta,b}f(t) \tag{14}
$$

即时间尺度联合 R-L 导数和时间尺度联合 Caputo 导数分别退化为时间尺度 Riesz-R-L 和 Riesz-Caputo 导数. 当 $\mathbb{T}=\mathbb{R}$ 时,式(9)和式(10)分别退化为传统的联合 R-L 导数和联合 Caputo 导数

$$
{}^{\mathrm{RL}}D^{\alpha,\beta}_{\gamma}f(t)=\gamma{}^{\mathrm{RL}}_{a}D^{\alpha}_{t}f(t)+(-1)^{n}(1-\gamma){}^{\mathrm{RL}}_{t}D^{\beta}_{b}f(t) \tag{15}
$$
$$
{}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma}f(t)=\gamma{}^{\mathrm{C}}_{a}D^{\alpha}_{t}f(t)+(-1)^{n}(1-\gamma){}^{\mathrm{C}}_{t}D^{\beta}_{b}f(t) \tag{16}
$$

式(15)和式(16)可见文献[25].

时间尺度分数阶导数的分部积分公式为[42]

$$
\int_a^b g(t){}^{\mathrm{C}}_{a}D^{\alpha}_{\Delta,t}f(t)\Delta t=\int_a^b f^{\sigma}(t)\cdot{}^{\mathrm{RL}}_{t}D^{\alpha}_{\Delta,b}g(t)\Delta t+\left[f(t)\cdot{}_{t}I^{1-\alpha}_{\Delta,b}g(t)\right]\Big|_a^b \tag{17}
$$

和

$$
\int_a^b g(t){}^{\mathrm{C}}_{t}D^{\alpha}_{\Delta,b}f(t)\Delta t=\int_a^b f^{\sigma}(t)\cdot{}^{\mathrm{RL}}_{a}D^{\alpha}_{\Delta,t}g(t)\Delta t-\left[f(t)\cdot{}_{a}I^{1-\alpha}_{\Delta,t}g(t)\right]\Big|_a^b \tag{18}
$$

2　时间尺度联合 Caputo 分数阶运动微分方程

2.1　Lagrange 系统运动微分方程

时间尺度分数阶 Lagrange 函数为 $L(t,q^{\sigma}_{k},{}^{\mathrm{C}}D^{\alpha,\beta}_{\Delta,\gamma}q_{k})$,则 Hamilton 作用量可表示为

$$
S_L[q(\cdot)]=\int_a^b L(t,q^{\sigma}_{k},{}^{\mathrm{C}}D^{\alpha,\beta}_{\Delta,\gamma}q_{k})\Delta t \tag{19}
$$

其中 $q^{\sigma}_{k}(t)=(q_k\circ\sigma)(t),t\in[a,b],0<\alpha,\beta<1,\gamma\in[0,1]$.

时间尺度联合 Caputo 分数阶变分原理为

$$
\delta S_L=0,\delta{}^{\mathrm{C}}_{a}D^{\alpha}_{\Delta,t}q_k={}^{\mathrm{C}}_{a}D^{\alpha}_{\Delta,t}\delta q_k,
$$
$$
\delta{}^{\mathrm{C}}_{t}D^{\alpha}_{\Delta,b}q_k={}^{\mathrm{C}}_{t}D^{\alpha}_{\Delta,b}\delta q_k,(\delta q_k)^{\sigma}=\delta q^{\sigma}_{k},
$$
$$
\delta q_k(t)\big|_{t=a}=\delta q_k(t)\big|_{t=b}=0 \tag{20}
$$

其中 δ 表示等时变分.

把式(19)代入式(20)可得

$$
\delta S_L=\int_a^b(\partial_j L\cdot\delta q^{\sigma}_{j}+\partial_{n+j}L\cdot\delta{}^{\mathrm{C}}D^{\alpha,\beta}_{\Delta,\gamma}q_j)\Delta t=0 \tag{21}
$$

其中 $\partial_j L$ 表示对 Lagrange 函数 L 的第 j 个元素求偏导数,$\partial_{n+j}L$ 表示对 Lagrange 函数 L 的第 $n+j$ 个元素求偏导数,$j=1,2,\cdots,n$.

利用分部积分公式(17)和公式(18)可得

$$
\begin{aligned}
&\int_a^b\partial_{n+j}L\cdot\delta{}^{\mathrm{C}}D^{\alpha,\beta}_{\Delta,\gamma}q_j\Delta t\\
&=\int_a^b\partial_{n+j}L\cdot\delta[\gamma{}^{\mathrm{C}}_{a}D^{\alpha}_{\Delta,t}q_j+(-1)^{n}(1-\gamma){}^{\mathrm{C}}_{t}D^{\beta}_{\Delta,b}q_j]\Delta t\\
&=-\int_a^b\delta q^{\sigma}_{j}[(1-\gamma)_{a}D^{\beta}_{\Delta,t}\partial_{n+j}L-\gamma_{t}D^{\alpha}_{\Delta,b}\partial_{n+j}L]\Delta t\\
&=-\int_a^b(\delta q^{\sigma}_{j}\cdot{}^{\mathrm{RL}}D^{\beta,\alpha}_{\Delta,1-\gamma}\partial_{n+j}L)\Delta t
\end{aligned} \tag{22}
$$

将式(22)代入式(21)且利用式(20)中的边界条件可得

$$
\begin{aligned}
0=\delta S_L&=\int_a^b(\partial_j L\cdot\delta q^{\sigma}_{j}-\delta q^{\sigma}_{j}\cdot{}^{\mathrm{RL}}D^{\beta,\alpha}_{\Delta,1-\gamma}\partial_{n+j}L)\Delta t\\
&=\int_a^b\{[\int_a^t(\partial_j L-{}^{\mathrm{RL}}D^{\beta,\alpha}_{\Delta,1-\gamma}\partial_{n+j}L)\Delta\theta\cdot\delta q_j]^{\Delta}-\\
&\quad(\delta q_j)^{\Delta}\cdot\int_a^t(\partial_j L-{}^{\mathrm{RL}}D^{\beta,\alpha}_{\Delta,1-\gamma}\partial_{n+j}L)\Delta\theta\}\Delta t\\
&=-\int_a^b[\int_a^t(\partial_j L-{}^{\mathrm{RL}}D^{\beta,\alpha}_{\Delta,1-\gamma}\partial_{n+j}L)\Delta\theta\cdot(\delta q_j)^{\Delta}]\Delta t
\end{aligned} \tag{23}
$$

从而

$$
\int_a^t(\partial_j L-{}^{\mathrm{RL}}D^{\beta,\alpha}_{\Delta,1-\gamma}\partial_{n+j}L)\Delta\theta=\mathrm{const} \tag{24}
$$

式(24)两边同时对 t 求 Δ 导数,可得

$$
\partial_j L-{}^{\mathrm{RL}}D^{\beta,\alpha}_{\Delta,1-\gamma}\partial_{n+j}L=0 \tag{25}
$$

式(25)称为时间尺度联合 Caputo 分数阶 Lagrange 方程.

注 4　当 $\mathbb{T}=\mathbb{R}$ 时,式(25)退化为

$$
\partial_j L(t,q_k,{}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma}q_k)-{}^{\mathrm{RL}}D^{\beta,\alpha}_{1-\gamma}\partial_{n+j}L(t,q_k,{}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma}q_k)=0 \tag{26}
$$

式(26)为联合 Caputo 分数阶 Lagrange 方程,与文献[43]所得结果一致.

注 5　当 $\alpha\to1,\beta\to1$ 时,式(25)退化为

$$
\partial_j L(t,q^{\sigma}_{k},q^{\Delta}_{k})-[\partial_{n+j}L(t,q^{\sigma}_{k},q^{\Delta}_{k})]^{\Delta}=0 \tag{27}
$$

式(27)为时间尺度上的 Lagrange 方程,与文献[13]所得结果一致.

注6 当 $\gamma=1$ 时，式(25)退化为

$$\partial_j L(t,q_k^\sigma,{}_a^{\mathrm{C}}D_{\Delta,t}^{\alpha}q_k)+{}_t^{\mathrm{RL}}D_{\Delta,b}^{\alpha}\partial_{n+j}L(t,q_k^\sigma,{}_a^{\mathrm{C}}D_{\Delta,t}^{\alpha}q_k)=0 \tag{28}$$

式(28)为时间尺度左Caputo分数阶Lagrange方程，其与文献[38]中所得结果一致. 当 $\gamma=0$ 时，式(25)退化为

$$\partial_j L(t,q_k^\sigma,{}_t^{\mathrm{C}}D_{\Delta,b}^{\alpha}q_k)+{}_a^{\mathrm{RL}}D_{\Delta,t}^{\alpha}\partial_{n+j}L(t,q_k^\sigma,{}_t^{\mathrm{C}}D_{\Delta,b}^{\alpha}q_k)=0 \tag{29}$$

式(29)称为时间尺度右Caputo分数阶Lagrange方程. 当 $\alpha=\beta,\gamma=1/2$ 时，式(25)退化为

$$\partial_j L(t,q_k^\sigma,{}_a^{RC}D_{\Delta,b}^{\alpha}q_k)-{}_a^{R}D_{\Delta,b}^{\alpha}\partial_{n+j}L(t,q_k^\sigma,{}_a^{RC}D_{\Delta,b}^{\alpha}q_k)=0 \tag{30}$$

式(30)称为时间尺度Riesz-Caputo分数阶Lagrange方程.

2.2 Hamilton系统运动微分方程

时间尺度Caputo分数阶Lagrange函数为 $L(t,q_k^\sigma,{}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_k)$，则广义动量和Hamilton函数为

$$p_k=\frac{\partial L}{\partial\,{}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_k},$$

$$H=H(t,q_k^\sigma,p_k)=p_j\cdot{}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_j-L(t,q_k^\sigma,{}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_k) \tag{31}$$

Hamilton作用量可表示为

$$S_H=\int_a^b(p_j\cdot{}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_j-H)\Delta t \tag{32}$$

时间尺度联合Caputo分数阶变分原理为

$$\delta S_H=0,\ \delta\,{}_a^{\mathrm{C}}D_{\Delta,t}^{\alpha}q_k={}_a^{\mathrm{C}}D_{\Delta,t}^{\alpha}\delta q_k,\ \delta\,{}_t^{\mathrm{C}}D_{\Delta,b}^{\alpha}q_k={}_t^{\mathrm{C}}D_{\Delta,b}^{\alpha}\delta q_k,\ (\delta q_k)^\sigma=\delta q_k^\sigma,\ \delta q_k(t)|_{t=a}=\delta q_k(t)|_{t=b}=0 \tag{33}$$

又

$$\begin{aligned}\int_a^b p_j\cdot\delta\,{}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_j\Delta t&=\int_a^b p_j\cdot{}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}\delta q_j\Delta t\\&=\int_a^b p_j\cdot[\gamma\,{}_a^{\mathrm{C}}D_{\Delta,t}^{\alpha}\delta q_j-(1-\gamma)\,{}_t^{\mathrm{C}}D_{\Delta,b}^{\beta}\delta q_j]\Delta t\\&=\gamma[\int_a^b\delta q_j^\sigma\cdot{}_t^{\mathrm{RL}}D_{\Delta,b}^{\alpha}p_j\Delta t+(\delta q_j\cdot{}_tI_{\Delta,b}^{1-\alpha}p_j)|_{t=a}^{t=b}]-\\&\quad(1-\gamma)[\int_a^b\delta q_j^\sigma\cdot{}_a^{\mathrm{RL}}D_{\Delta,t}^{\beta}p_j\Delta t-(\delta q_j\cdot{}_aI_{\Delta,t}^{1-\beta}p_j)|_{t=a}^{t=b}]\\&=-\int_a^b(\delta q_j^\sigma\cdot{}^{\mathrm{RL}}D_{\Delta,1-\gamma}^{\beta,\alpha}p_j)\Delta t\end{aligned} \tag{34}$$

故

$$\begin{aligned}0=\delta S_H&=\int_a^b(\delta p_j\cdot{}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_j+p_j\cdot\delta\,{}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_j-\partial_jH\cdot\delta q_j^\sigma-\partial_{n+j}H\cdot\delta p_j)\Delta t\\&=\int_a^b(\delta p_j\cdot{}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_j-\delta q_j^\sigma\,{}^{\mathrm{RL}}D_{\Delta,1-\gamma}^{\beta,\alpha}p_j-\partial_jH\cdot\delta q_j^\sigma-\partial_{n+j}H\cdot\delta p_j)\Delta t\\&=\int_a^b[\delta p_j\cdot({}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_j-\partial_{n+j}H)-\delta q_j^\sigma({}^{\mathrm{RL}}D_{\Delta,1-\gamma}^{\beta,\alpha}p_j+\partial_jH)]\Delta t\end{aligned} \tag{35}$$

将式(31)中的Hamilton函数两边对 p_j 求导，可得

$$\partial_{n+j}H={}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_j \tag{36}$$

将式(36)代入式(35)可得

$$\begin{aligned}0=\delta S_H&=-\int_a^b\delta q_j^\sigma({}^{\mathrm{RL}}D_{\Delta,1-\gamma}^{\beta,\alpha}p_j+\partial_jH)\Delta t\\&=-\int_a^b\{[\int_a^t({}^{\mathrm{RL}}D_{\Delta,1-\gamma}^{\beta,\alpha}p_j+\partial_jH)\Delta\theta\cdot\delta q_j]^\Delta-(\delta q_j)^\Delta\int_a^t({}^{\mathrm{RL}}D_{\Delta,1-\gamma}^{\beta,\alpha}p_j+\partial_jH)\Delta\theta\}\Delta t\\&=\int_a^b[(\delta q_j)^\Delta\cdot\int_a^t({}^{\mathrm{RL}}D_{\Delta,1-\gamma}^{\beta,\alpha}p_j+\partial_jH)\Delta\theta]\Delta t\end{aligned} \tag{37}$$

从而

$$\int_a^t({}^{\mathrm{RL}}D_{\Delta,1-\gamma}^{\beta,\alpha}p_j+\partial_jH)\Delta\theta=\mathrm{const} \tag{38}$$

即

$${}^{\mathrm{RL}}D_{\Delta,1-\gamma}^{\beta,\alpha}p_j+\partial_jH=0 \tag{39}$$

将式(36)与式(39)放在一起可得

$$\left.\begin{aligned}\partial_jH&=-{}^{\mathrm{RL}}D_{\Delta,1-\gamma}^{\beta,\alpha}p_j\\\partial_{n+j}H&={}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_j\end{aligned}\right\} \tag{40}$$

式(40)称为时间尺度联合Caputo分数阶Hamilton方程.

注7 当 $\mathbb{T}=\mathbb{R}$ 时，$H=H(t,q_k,p_k)$ 时，式(40)退化为

$$\partial_jH=-{}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}p_j,\ \partial_{n+j}H={}^{\mathrm{C}}D_{\gamma}^{\alpha,\beta}q_j \tag{41}$$

式(41)为联合Caputo分数阶Hamilton方程，其与文献[43]中所得结果一致.

注8 当 $\alpha\to1,\beta\to1$ 时，$H=H(t,q_k^\sigma,p_k)$，此时式(40)退化为

$$\partial_jH=-p_j^\Delta,\ \partial_{n+j}H=q_j^\Delta \tag{42}$$

式(42)为时间尺度上的Hamilton方程，其与文献[18]中所得结果一致.

注9 当 $\gamma=1$ 时，方程(40)退化为

$$\partial_jH(t,q_k^\sigma,p_k)={}_t^{\mathrm{RL}}D_{\Delta,b}^{\alpha}p_j,\ \partial_{n+j}H(t,q_k^\sigma,p_k)={}_a^{\mathrm{C}}D_{\Delta,t}^{\alpha}q_j \tag{43}$$

式(43)为时间尺度左Caputo分数阶Hamilton方程，其与文献[39]所得结果一致. 当 $\gamma=0$ 时，式(40)退化为

$$\partial_j H(t,q_k^\sigma,p_k)=-{}_a^{RL}D_{\Delta,t}^{\alpha}p_j\ ,$$
$$\partial_{n+j} H(t,q_k^\sigma,p_k)=-{}_t^{C}D_{\Delta,b}^{\alpha}q_j \tag{44}$$

式(44)称为时间尺度右 Caputo 分数阶 Hamilton 方程. 当 $\alpha=\beta,\gamma=1/2$ 时,方程(40)退化为

$$\partial_j H(t,q_k^\sigma,p_k)=-{}_a^{R}D_{\Delta,b}^{\alpha}p_j\ ,$$
$$\partial_{n+j} H(t,q_k^\sigma,p_k)={}_a^{RC}D_{\Delta,b}^{\alpha}q_j \tag{45}$$

式(45)称为时间尺度 Riesz-Caputo 分数阶 Hamilton 方程.

3 时间尺度联合 Caputo 分数阶 Noether 定理

3.1 Lagrange 系统 Noether 定理

定义 3 对于一个物理量 I,如果满足 $\Delta I/\Delta t=0$,则称 I 为一个守恒量.

引入无限小变换

$$\bar{t}=T_{L\varepsilon}=t+\varepsilon\xi_{L0}+o(\varepsilon)\ ,$$
$$\bar{q}_j=Q_{L\varepsilon}^j=q_j+\varepsilon\xi_{Lj}+o(\varepsilon)\ ,\ j=1,2,\cdots,n \tag{46}$$

其中 ξ_{L0}、ξ_{Lj} 是无限小生成元,ε 是无限小参数. 设映射 $t\mapsto\nu(t):T_{L\varepsilon}(t,q_k)\in\mathbb{R}$ 为递增的,映射的像构成一个新的时间尺度 $\bar{T}$,其前跳算子为 $\bar{\sigma}$,对应导数为 $\bar{\Delta}$,且满足 $\sigma\circ\nu=\nu\circ\sigma$.

Hamilton 作用量的不变性可表示为,对于任意的 $[t_a,t_b]\subset[a,b]$,恒有

$$\int_{t_a}^{t_b}L[t,q_k^\sigma(t),{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_k(t)]\Delta t$$
$$=\int_{\bar{t}_a}^{\bar{t}_b}\bar{L}[\bar{t},\bar{q}_k^{\bar{\sigma}}(\bar{t}),{}^{C}D_{\bar{\Delta},\gamma}^{\alpha,\beta}\bar{q}_k(\bar{t})]\bar{\Delta}\bar{t} \tag{47}$$

成立. 即

$$\int_{t_a}^{t_b}L[t,q_k^\sigma(t),{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_k(t)]\Delta t$$
$$=\int_{t_a}^{t_b}L\left[T_{L\varepsilon},(Q_{L\varepsilon}^k)^\sigma,\frac{{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}Q_{L\varepsilon}^k}{(T_{L\varepsilon}^{\Delta})^\alpha}\right]\cdot T_{L\varepsilon}^{\Delta}\Delta t \tag{48}$$

其中

$${}^{C}D_{\bar{\Delta},\gamma}^{\alpha,\beta}\bar{q}_k(\bar{t})=\frac{{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}(\bar{q}_k\circ v)(t)}{[v^{\Delta}(t)]^\alpha} \tag{49}$$

式(48)等价于

$$L[t,q_k^\sigma(t),{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_k(t)]$$
$$=L\left[T_{L\varepsilon},(Q_{L\varepsilon}^k)^\sigma,\frac{{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}Q_{L\varepsilon}^k}{(T_{L\varepsilon}^{\Delta})^\alpha}\right]\cdot T_{L\varepsilon}^{\Delta} \tag{50}$$

即

$$0=\partial_0 L\cdot\xi_{L0}+\partial_k L\cdot\xi_{Lk}^\sigma+\partial_{k+n}L\cdot\left.\frac{{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}\xi_{Lk}-\alpha\,{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_k\cdot\xi_{L0}^{\Delta}}{(T_{L\varepsilon}^{\Delta})^{2\alpha}}\right|_{\varepsilon=0}+L\xi_{L0}^{\Delta}$$
$$=\partial_0 L\cdot\xi_{L0}+\partial_k L\cdot\xi_{Lk}^\sigma+\partial_{k+n}L\cdot({}^{C}D_{\Delta,\gamma}^{\alpha,\beta}\xi_{Lk}-\alpha\,{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_k\cdot\xi_{L0}^{\Delta})+L\xi_{L0}^{\Delta} \tag{51}$$

从而 Noether 恒等式为

$$\partial_0 L\cdot\xi_{L0}+\partial_k L\cdot\xi_{Lk}^\sigma+\partial_{k+n}L\cdot({}^{C}D_{\Delta,\gamma}^{\alpha,\beta}\xi_{Lk}-\alpha\,{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_k\cdot\xi_{L0}^{\Delta})+L\xi_{L0}^{\Delta}=0 \tag{52}$$

定理 1 对于时间尺度联合 Caputo 分数阶 Lagrange 系统[式(25)],如果 ξ_{L0} 和 ξ_{Lj} 满足 Noether 恒等式[式(52)],则该系统存在如下守恒量

$$I_L=\int_a^t[\partial_0 L+(\alpha\cdot\partial_{n+k}L\cdot{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_k+\partial_0 L\cdot\mu(\tau)-L)^{\Delta}]\cdot\xi_{L0}^\sigma\Delta\tau+[L-\alpha\cdot\partial_{n+k}L\cdot{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_k-\partial_0 L\cdot\mu(t)]\cdot\xi_{L0}+\int_a^t({}^{RL}D_{\Delta,1-\gamma}^{\beta,\alpha}\partial_{n+k}L\cdot\xi_{Lk}^\sigma+\partial_{n+k}L\cdot{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}\xi_{Lk})\Delta\tau \tag{53}$$

证明 利用式(25)和式(52)可得

$$\frac{\Delta}{\Delta t}I_L=\partial_0 L\cdot\xi_{L0}^\sigma+[\alpha\cdot\partial_{n+k}L\cdot{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_k+\partial_0 L\cdot\mu(t)-L]^{\Delta}\cdot\xi_{L0}^\sigma+[L-\alpha\cdot\partial_{n+k}L\cdot{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_k-\partial_0 L\cdot\mu(t)]\cdot\xi_{L0}^{\Delta}+[L-\alpha\cdot\partial_{n+k}L\cdot{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_k-\partial_0 L\cdot\mu(t)]^{\Delta}\cdot\xi_{L0}^\sigma+{}^{RL}D_{\Delta,1-\gamma}^{\beta,\alpha}\partial_{n+k}L\cdot\xi_{Lk}^\sigma+\partial_{n+k}L\cdot{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}\xi_{Lk}=-\partial_k L\cdot\xi_{Lk}^\sigma+{}^{RL}D_{\Delta,1-\gamma}^{\beta,\alpha}\partial_{n+k}L\cdot\xi_{Lk}^\sigma=0$$

证毕.

注 10 当 $\mathbb{T}=\mathbb{R}$ 时,式(52)退化为

$$\partial_0 L\cdot\xi_{L0}+\partial_k L\cdot\xi_{Lk}+\partial_{k+n}L\cdot({}^{C}D_{\gamma}^{\alpha,\beta}\xi_{Lk}-\alpha\,{}^{C}D_{\gamma}^{\alpha,\beta}q_k\cdot\dot{\xi}_{L0})+L\dot{\xi}_{L0}=0 \tag{54}$$

式(53)退化为

$$I_L=\int_a^t\left[\partial_0 L+\frac{d}{dt}(\alpha\cdot\partial_{n+k}L\cdot{}^{C}D_{\gamma}^{\alpha,\beta}q_k-L)\right]\cdot\xi_{L0}d\tau+(L-\alpha\cdot\partial_{n+k}L\cdot{}^{C}D_{\gamma}^{\alpha,\beta}q_k)\cdot\xi_{L0}+\int_a^t({}^{RL}D_{1-\gamma}^{\beta,\alpha}\partial_{n+k}L\cdot\xi_{Lk}+\partial_{n+k}L\cdot{}^{C}D_{\gamma}^{\alpha,\beta}\xi_{Lk})d\tau \tag{55}$$

定理 2 对于联合 Caputo 分数阶 Lagrange 系统[式(26)],如果 ξ_{L0} 和 ξ_{Lj} 满足式(54),则该系统存在如式(55)的守恒量.

证明 利用式(26)和式(54)可得

$$\frac{d}{dt}I_L=\partial_0 L\cdot\xi_{L0}+\frac{d}{dt}(\alpha\cdot\partial_{n+k}L\cdot{}^{C}D_{\gamma}^{\alpha,\beta}q_k-L)\cdot\xi_{L0}+(L-\alpha\cdot\partial_{n+k}L\cdot{}^{C}D_{\gamma}^{\alpha,\beta}q_k)\dot{\xi}_{L0}+\frac{d}{dt}(L-\alpha\cdot\partial_{n+k}L\cdot{}^{C}D_{\gamma}^{\alpha,\beta}q_k)\cdot\xi_{L0}+{}^{RL}D_{1-\gamma}^{\beta,\alpha}\partial_{n+k}L\cdot\xi_{Lk}+\partial_{n+k}L\cdot{}^{C}D_{\gamma}^{\alpha,\beta}\xi_{Lk}=(-\partial_k L+{}^{RL}D_{1-\gamma}^{\beta,\alpha}\partial_{n+k}L)\cdot$$

$$\xi_{Lk}=0$$

证毕.

注 11 当 $\alpha\to1,\beta\to1$ 时,式(52)退化为

$$\partial_0 L\cdot\xi_{L0}+\partial_k L\cdot\xi_{Lk}^{\sigma}+\partial_{k+n}L\cdot(\xi_{Lk}^{\Delta}-q_k^{\Delta}\cdot\xi_{L0}^{\Delta})+L\xi_{L0}^{\Delta}=0 \tag{56}$$

式(53)退化为

$$I_L=\int_a^t\{\partial_0 L+[\partial_{n+k}L\cdot q_k^{\Delta}+\partial_0 L\cdot\mu(\tau)-L]^{\Delta}\}\cdot\xi_{L0}^{\sigma}\Delta\tau+[L-\partial_{n+k}L\cdot q_k^{\Delta}-\partial_0 L\cdot\mu(t)]\cdot\xi_{L0}+\partial_{n+k}L\cdot\xi_{Lk} \tag{57}$$

定理 3 对于时间尺度上的 Lagrange 系统[式(27)],如果 ξ_{L0} 和 ξ_{Lj} 满足式(56),则该系统存在如式(57)的守恒量.

证明 利用式(27)和式(56)可得 $\Delta I_L/\Delta t=0$.

证毕.此定理与文献[44]所得结果一致.

注 12 当 $\gamma=1$ 时,可得时间尺度左 Caputo 分数阶 Lagrange 系统[式(28)],其 Noether 恒等式为

$$\partial_0 L\cdot\xi_{L0}+\partial_k L\cdot\xi_{Lk}^{\sigma}+\partial_{k+n}L\cdot({}_a^{C}D_{\Delta,t}^{\alpha}\xi_{Lk}-\alpha\,{}_a^{C}D_{\Delta,t}^{\alpha}q_k\cdot\xi_{L0}^{\Delta})+L\xi_{L0}^{\Delta}=0 \tag{58}$$

对应 Noether 守恒量为

$$I_L=\int_a^t\{\partial_0 L+[\alpha\cdot\partial_{n+k}L\cdot{}_a^{C}D_{\Delta,t}^{\alpha}q_k+\partial_0 L\cdot\mu(\tau)-L]^{\Delta}\}\cdot\xi_{L0}^{\sigma}\Delta\tau+[L-\alpha\cdot\partial_{n+k}L\cdot{}_a^{C}D_{\Delta,t}^{\alpha}q_k-\partial_0 L\cdot\mu(t)]\cdot\xi_{L0}+\int_a^t(-{}_t^{RL}D_{\Delta,b}^{\alpha}\partial_{n+k}L\cdot\xi_{Lk}^{\sigma}+\partial_{n+k}L\cdot{}_a^{C}D_{\Delta,t}^{\alpha}\xi_{Lk})\Delta\tau \tag{59}$$

此结果与文献[38]所得结果一致.

注 13 当 $\gamma=0$ 时,可得时间尺度右 Caputo 分数阶 Lagrange 系统[式(29)],其 Noether 恒等式为

$$\partial_0 L\cdot\xi_{L0}+\partial_k L\cdot\xi_{Lk}^{\sigma}+\partial_{k+n}L\cdot(-{}_t^{C}D_{\Delta,b}^{\alpha}\xi_{Lk}+\alpha\,{}_t^{C}D_{\Delta,b}^{\alpha}q_k\cdot\xi_{L0}^{\Delta})+L\xi_{L0}^{\Delta}=0 \tag{60}$$

对应 Noether 守恒量为

$$I_L=\int_a^t\{\partial_0 L+[-\alpha\cdot\partial_{n+k}L\cdot{}_t^{C}D_{\Delta,b}^{\alpha}q_k+\partial_0 L\cdot\mu(\tau)-L]^{\Delta}\}\cdot\xi_{L0}^{\sigma}\Delta\tau+[L+\alpha\cdot\partial_{n+k}L\cdot{}_t^{C}D_{\Delta,b}^{\alpha}q_k-\partial_0 L\cdot\mu(t)]\cdot\xi_{L0}+\int_a^t({}_a^{RL}D_{\Delta,t}^{\alpha}\partial_{n+k}L\cdot\xi_{Lk}^{\sigma}-\partial_{n+k}L\cdot{}_t^{C}D_{\Delta,b}^{\alpha}\xi_{Lk})\Delta\tau \tag{61}$$

注 14 当 $\alpha=\beta,\gamma=1/2$ 时,可得时间尺度 Riesz-Caputo 分数阶 Lagrange 系统[式(30)],其 Noether 恒等式为

$$\partial_0 L\cdot\xi_{L0}+\partial_k L\cdot\xi_{Lk}^{\sigma}+\partial_{k+n}L\cdot({}_a^{C}D_{\Delta,b}^{\alpha}\xi_{Lk}-\alpha\,{}_a^{C}D_{\Delta,b}^{\alpha}q_k\cdot\xi_{L0}^{\Delta})+L\xi_{L0}^{\Delta}=0 \tag{62}$$

对应 Noether 守恒量为

$$I_L=\int_a^t\{\partial_0 L+[\alpha\cdot\partial_{n+k}L\cdot{}_a^{C}D_{\Delta,b}^{\alpha}q_k+\partial_0 L\cdot\mu(\tau)-L]^{\Delta}\}\cdot\xi_{L0}^{\sigma}\Delta\tau+[L-\alpha\cdot\partial_{n+k}L\cdot{}_a^{C}D_{\Delta,b}^{\alpha}q_k-\partial_0 L\cdot\mu(t)]\cdot\xi_{L0}+\int_a^t({}_a^{RL}D_{\Delta,b}^{\alpha}\partial_{n+k}L\cdot\xi_{Lk}^{\sigma}+\partial_{n+k}L\cdot{}_a^{C}D_{\Delta,b}^{\alpha}\xi_{Lk})\Delta\tau \tag{63}$$

3.2 Hamilton 系统 Noether 定理

取无限小变换为

$$\bar{t}=T_{H\varepsilon}=t+\varepsilon\xi_{H0}+o(\varepsilon),\quad \bar{q}_j=Q_{H\varepsilon}^j=q_j+\varepsilon\xi_{Hj}+o(\varepsilon),\quad \bar{p}_j=P_{H\varepsilon}^j=p_j+\varepsilon\eta_{Hj}+o(\varepsilon) \tag{64}$$

其中 $\xi_{H0},\xi_{Hj},\eta_{Hj},j=1,2,\cdots,n$ 是无限小生成元,ε 是无限小参数.设映射 $t\mapsto\nu(t):T_{H\varepsilon}(t,q_k,p_k)\in\mathbb{R}$ 为递增的,映射的像构成一个新的时间尺度 $\bar{T}$,其前跳算子为 $\bar{\sigma}$,对应导数为 $\bar{\Delta}$,且满足 $\sigma\circ\nu=\nu\circ\sigma$.

Hamilton 作用量的不变性可表示为对于任意的 $[t_a,t_b]\subset[a,b]$,恒有

$$\int_{t_a}^{t_b}\{p_j(t)\cdot{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_j(t)-H[t,q_k^{\sigma}(t),p_k(t)]\}\Delta t=\int_{\bar{t}_a}^{\bar{t}_b}\{\bar{p}_j(\bar{t})\cdot{}^{C}D_{\bar{\Delta},\gamma}^{\alpha,\beta}\bar{q}_j(\bar{t})-\bar{H}[\bar{t},\bar{q}_k^{\bar{\sigma}}(\bar{t}),\bar{p}_k(\bar{t})]\}\bar{\Delta}\bar{t} \tag{65}$$

成立.利用式(49)可得

$$\int_{t_a}^{t_b}\{p_j(t)\cdot{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_j(t)-H[t,q_k^{\sigma}(t),p_k(t)]\}\Delta t=\int_{t_a}^{t_b}\left\{P_{H\varepsilon}^j\cdot\frac{{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}Q_{H\varepsilon}^j}{(T_{H\varepsilon}^{\Delta})^{\alpha}}-H[T_{H\varepsilon},(Q_{H\varepsilon}^k)^{\sigma},P_{H\varepsilon}^k]\right\}T_{H\varepsilon}^{\Delta}\Delta t \tag{66}$$

由 $[t_a,t_b]$ 的任意性可得

$$p_j(t)\cdot{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_j(t)-H[t,q_k^{\sigma}(t),p_k(t)]=\left\{P_{H\varepsilon}^j\cdot\frac{{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}Q_{H\varepsilon}^j}{(T_{H\varepsilon}^{\Delta})^{\alpha}}-H[T_{H\varepsilon},(Q_{H\varepsilon}^k)^{\sigma},P_{H\varepsilon}^k]\right\}T_{H\varepsilon}^{\Delta} \tag{67}$$

即

$$p_j\cdot{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}\xi_{Hj}-(\alpha-1)p_j\cdot\xi_{H0}^{\Delta}\,{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_j-H\xi_{H0}^{\Delta}-\partial_0 H\cdot\xi_{H0}-\partial_k H\cdot\xi_{Hk}^{\sigma}=0 \tag{68}$$

式(68)称为时间尺度联合 Caputo 分数阶 Noether 恒等式.

定理 4 对于时间尺度联合 Caputo 分数阶 Hamilton 系统[式(40)],若无限小生成元满足式(68),则该系统存在如下形式的守恒量

$$I_H=\int_a^t\{[H+(\alpha-1)p_k\cdot\partial_{n+k}H-\partial_0H\cdot\mu(\tau)]^\Delta-\partial_0H\}\cdot\xi_{H0}^\sigma\Delta\tau-[H+(\alpha-1)p_k\cdot\partial_{n+k}H-\partial_0H\cdot\mu(t)]\cdot\xi_{H0}+\int_a^t(p_k\cdot{}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}\xi_{Hk}+\xi_{Hk}^\sigma\cdot{}^{\mathrm{RL}}D_{\Delta,1-\gamma}^{\beta,\alpha}p_k)\Delta\tau \tag{69}$$

证明　利用式(40)和式(68)可得

$$\frac{\Delta I_H}{\Delta t}=[H+(\alpha-1)p_k\cdot\partial_{n+k}H-\partial_0H\cdot\mu(\tau)]^\Delta\xi_{H0}^\sigma-\partial_0H\cdot\xi_{H0}^\sigma-[H+(\alpha-1)p_k\cdot\partial_{n+k}H-\partial_0H\cdot\mu(t)]^\Delta\cdot\xi_{H0}^\sigma-[H+(\alpha-1)p_k\cdot\partial_{n+k}H-\partial_0H\cdot\mu(t)]\cdot\xi_{H0}^\Delta+p_k\cdot{}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}\xi_{Hk}+\xi_{Hk}^\sigma\cdot{}^{\mathrm{RL}}D_{\Delta,1-\gamma}^{\beta,\alpha}p_k$$
$$=\partial_kH\cdot\xi_{Hk}^\sigma+\xi_{Hk}^\sigma\cdot{}^{\mathrm{RL}}D_{\Delta,1-\gamma}^{\beta,\alpha}p_k$$
$$=\xi_{Hk}^\sigma(\partial_kH+{}^{\mathrm{RL}}D_{\Delta,1-\gamma}^{\beta,\alpha}p_k)=0$$

证毕.

注 15　当 $\mathbb{T}=\mathbb{R}$ 时,式(68)退化为

$$p_j\cdot{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}\xi_{Hj}-(\alpha-1)p_j\cdot\dot{\xi}_{H0}{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}q_j-H\dot{\xi}_{H0}-\partial_0H\cdot\xi_{H0}-\partial_kH\cdot\xi_{Hk}=0 \tag{70}$$

式(69)退化为

$$I_H=\int_a^t\{\frac{\mathrm{d}}{\mathrm{d}t}[H+(\alpha-1)p_k\cdot\partial_{n+k}H-\partial_0H\cdot\mu(\tau)]-\partial_0H\}\cdot\xi_{H0}\mathrm{d}\tau-[H+(\alpha-1)p_k\cdot\partial_{n+k}H-\partial_0H\cdot\mu(t)]\cdot\xi_{H0}+\int_a^t(p_k\cdot{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}\xi_{Hk}+\xi_{Hk}\cdot{}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}p_k)\mathrm{d}\tau \tag{71}$$

定理 5　对于联合 Caputo 分数阶 Hamilton 系统[式(41)],若无限小生成元满足式(70),则该系统存在形如式(71)的守恒量.

注 16　当 $\alpha\to1,\beta\to1$ 时,对于时间尺度上的 Hamilton 系统[式(42)],其 Noether 恒等式为

$$p_j\cdot\xi_{Hj}^\Delta-H\xi_{H0}^\Delta-\partial_0H\xi_{H0}-\partial_kH\cdot\xi_{Hk}^\sigma=0 \tag{72}$$

对应 Noether 守恒量为

$$I_H=\int_a^t\{[H-\partial_0H\cdot\mu(\tau)]^\Delta-\partial_0H\}\cdot\xi_{H0}^\sigma\Delta\tau-[H-\partial_0H\cdot\mu(t)]\cdot\xi_{H0}+p_k\cdot\xi_{Hk} \tag{73}$$

此结论与文献[45]所得结果一致.

注 17　当 $\gamma=1$ 时,对于时间尺度左 Caputo 分数阶 Hamilton 系统[式(43)],其 Noether 恒等式为

$$p_j\cdot{}_a^{\mathrm{C}}D_{\Delta,t}^\alpha\xi_{Hj}-(\alpha-1)p_j\cdot\xi_{H0}^\Delta{}_a^{\mathrm{C}}D_{\Delta,t}^\alpha q_j-H\xi_{H0}^\Delta-\partial_0H\cdot\xi_{H0}-\partial_kH\cdot\xi_{Hk}^\sigma=0 \tag{74}$$

对应 Noether 守恒量为

$$I_H=\int_a^t\{[H+(\alpha-1)p_k\cdot\partial_{n+k}H-\partial_0H\cdot\mu(\tau)]^\Delta-\partial_0H\}\cdot\xi_{H0}^\sigma\Delta\tau-[H+(\alpha-1)p_k\cdot\partial_{n+k}H-\partial_0H\cdot\mu(t)]\cdot\xi_{H0}+\int_a^t(p_k\cdot{}_a^{\mathrm{C}}D_{\Delta,t}^\alpha\xi_{Hk}-\xi_{Hk}^\sigma\cdot{}_t^{\mathrm{RL}}D_{\Delta,b}^\alpha p_k)\Delta\tau \tag{75}$$

此结果与文献[39]所得结果一致.

注 18　当 $\gamma=0$ 时,对于时间尺度右 Caputo 分数阶 Hamilton 系统[式(44)],其 Noether 恒等式为

$$-p_j\cdot{}_t^{\mathrm{C}}D_{\Delta,b}^\alpha\xi_{Hj}+(\alpha-1)p_j\cdot\xi_{H0}^\Delta{}_t^{\mathrm{C}}D_{\Delta,b}^\alpha q_j-H\xi_{H0}^\Delta-\partial_0H\cdot\xi_{H0}-\partial_kH\cdot\xi_{Hk}^\sigma=0 \tag{76}$$

对应 Noether 守恒量为

$$I_H=\int_a^t\{[H+(\alpha-1)p_k\cdot\partial_{n+k}H-\partial_0H\cdot\mu(\tau)]^\Delta-\partial_0H\}\cdot\xi_{H0}^\sigma\Delta\tau-[H+(\alpha-1)p_k\cdot\partial_{n+k}H-\partial_0H\cdot\mu(t)]\cdot\xi_{H0}-\int_a^t(p_k\cdot{}_t^{\mathrm{C}}D_{\Delta,b}^\alpha\xi_{Hk}-\xi_{Hk}^\sigma\cdot{}_a^{\mathrm{RL}}D_{\Delta,t}^\alpha p_k)\Delta\tau \tag{77}$$

注 19　当 $\alpha=\beta,\gamma=1/2$ 时,对于时间尺度 Riesz-Caputo 分数阶 Hamilton 系统[式(45)],其 Noether 恒等式为

$$p_j\cdot{}_a^{\mathrm{C}}D_{\Delta,b}^\alpha\xi_{Hj}-(\alpha-1)p_j\cdot\xi_{H0}^\Delta{}_a^{\mathrm{C}}D_{\Delta,b}^\alpha q_j-H\xi_{H0}^\Delta-\partial_0H\cdot\xi_{H0}-\partial_kH\cdot\xi_{Hk}^\sigma=0 \tag{78}$$

对应 Noether 守恒量为

$$I_H=\int_a^t\{[H+(\alpha-1)p_k\cdot\partial_{n+k}H-\partial_0H\cdot\mu(\tau)]^\Delta-\partial_0H\}\cdot\xi_{H0}^\sigma\Delta\tau-[H+(\alpha-1)p_k\cdot\partial_{n+k}H-\partial_0H\cdot\mu(t)]\cdot\xi_{H0}+\int_a^t(p_k\cdot{}_a^{\mathrm{C}}D_{\Delta,b}^\alpha\xi_{Hk}+\xi_{Hk}^\sigma\cdot{}_a^{\mathrm{RL}}D_{\Delta,b}^\alpha p_k)\Delta\tau \tag{79}$$

4　算例

例 1　研究分数阶 Kepler 问题.时间尺度联合 Caputo 分数阶 Lagrange 函数为

$$L(q_1^\sigma,q_2^\sigma,{}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_1,{}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_2)=\frac{1}{2}[({}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_1)^2+({}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_2)^2]+[(q_1^\sigma)^2+(q_2^\sigma)^2]^{-\frac{1}{2}} \tag{80}$$

考虑该系统的 Noether 对称性问题.

根据式(25)可得运动微分方程为

$$q_1^\sigma[(q_1^\sigma)^2+(q_2^\sigma)^2]^{-\frac{3}{2}}+{}^{\mathrm{RL}}D_{\Delta,1-\gamma}^{\beta,\alpha}{}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_1=0,$$
$$q_2^\sigma[(q_1^\sigma)^2+(q_2^\sigma)^2]^{-\frac{3}{2}}+{}^{\mathrm{RL}}D_{\Delta,1-\gamma}^{\beta,\alpha}{}^{\mathrm{C}}D_{\Delta,\gamma}^{\alpha,\beta}q_2=0 \tag{81}$$

根据式(52)可得 Noether 恒等式

$$-\frac{1}{2}[(q_1^\sigma)^2+(q_2^\sigma)^2]^{-\frac{3}{2}}\cdot2q_1^\sigma\cdot\xi_{L1}^\sigma-$$

$$\frac{1}{2}[(q_1^\sigma)^2+(q_2^\sigma)^2]^{-\frac{3}{2}}\cdot 2q_2^\sigma\cdot\xi_{L2}^\sigma+{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_1\cdot({}^{C}D_{\Delta,\gamma}^{\alpha,\beta}\xi_{L1}-\alpha\,{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_1\cdot\xi_{L0}^\Delta)+{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_2\cdot({}^{C}D_{\Delta,\gamma}^{\alpha,\beta}\xi_{L2}-\alpha\,{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_2\cdot\xi_{L0}^\Delta)+\{\frac{1}{2}[({}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_1)^2+({}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_2)^2]+[(q_1^\sigma)^2+(q_2^\sigma)^2]^{-\frac{1}{2}}\}\xi_{L0}^\Delta=0 \tag{82}$$

由式(82)可得解

$$\xi_{L0}=0,\xi_{L1}=-q_2,\xi_{L2}=q_1 \tag{83}$$

其对应守恒量为

$$I_L=\int_a^t(-q_2^\sigma\cdot{}^{RL}D_{\Delta,1-\gamma}^{\beta,\alpha}\,{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_1+q_1^\sigma\cdot{}^{RL}D_{\Delta,1-\gamma}^{\beta,\alpha}\,{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q_2)\Delta\tau=\text{const.} \tag{84}$$

当 $\gamma=1$ 时

$$I_L=\int_a^t(q_2^\sigma\cdot{}_{\tau}^{RL}D_{\Delta,b}^{\alpha}\,{}_{a}^{C}D_{\Delta,\tau}^{\alpha}q_1-q_1^\sigma\cdot{}_{\tau}^{RL}D_{\Delta,b}^{\alpha}\,{}_{a}^{C}D_{\Delta,\tau}^{\alpha}q_2)\Delta\tau=\text{const} \tag{85}$$

此结果与文献[38]所得结果一致.

当 $\alpha\to1,\beta\to1$ 时

$$I_L=\int_a^t(-q_2^\sigma\cdot q_1^{\Delta\Delta}+q_1^\sigma\cdot q_2^{\Delta\Delta})\Delta\tau=-q_2q_1^\Delta+q_1q_2^\Delta=\text{const} \tag{86}$$

此结果与文献[15]所得结果一致.

例2 研究单自由度线性分数阶振动系统的Noether对称性,其中Lagrange函数为

$$L=\frac{1}{2}e^{\zeta t}[({}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q)^2-(q^\sigma)^2] \tag{87}$$

时间尺度取 $\mathbb{T}=\{a+mh,m\in\mathbb{N}\}$.

由式(87)可得广义动量和Hamilton函数为

$$p=e^{\zeta t}\cdot{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q\ ,\ H=\frac{p^2}{2e^{\zeta t}}+\frac{1}{2}e^{\zeta t}(q^\sigma)^2 \tag{88}$$

根据式(40)可得运动微分方程为

$${}^{RL}D_{\Delta,1-\gamma}^{\beta,\alpha}p=-e^{\zeta t}\cdot q^\sigma\ ,\ {}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q=\frac{2p}{2e^{\zeta t}}=\frac{p}{e^{\zeta t}} \tag{89}$$

根据式(68)可得Noether恒等式

$$p\cdot{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}\xi_H-(\alpha-1)p\cdot\xi_{H0}^\Delta\cdot{}^{C}D_{\Delta,\gamma}^{\alpha,\beta}q-\left[\frac{p^2}{2e^{\zeta t}}+\frac{1}{2}e^{\zeta t}(q^\sigma)^2\right]\xi_{H0}^\Delta-\left[\frac{1}{2}e^{\zeta t}(q^\sigma)^2\zeta+\frac{1}{2}p^2\cdot\frac{-\zeta}{e^{\zeta t}}\right]\cdot\xi_{H0}-e^{\zeta t}q^\sigma\cdot\xi_H^\sigma=0 \tag{90}$$

由式(90)可得解

$$\xi_{H0}=1,\ \xi_H=-\frac{\zeta}{2}q \tag{91}$$

其对应守恒量为

$$I_H=\frac{1}{2}\int_a^t\left[\frac{\zeta p^2}{e^{\zeta\tau}}-\zeta(q^\sigma)^2e^{\zeta\tau}\right]d\tau-\frac{\zeta q}{2}[\gamma_tI_{\Delta,b}^{1-\alpha}p+(1-\gamma)_aI_{\Delta,t}^{1-\beta}p] \tag{92}$$

当 $\gamma=1$ 时

$$I_H=\frac{1}{2}\int_a^t\left[\frac{\zeta p^2}{e^{\zeta\tau}}-\zeta(q^\sigma)^2e^{\zeta\tau}\right]d\tau-\frac{\zeta q}{2}\,{}_tI_{\Delta,b}^{1-\alpha}p=\text{const.} \tag{93}$$

此结果与文献[39]所得结果一致.

5 结论

文中讨论了时间尺度联合Caputo分数阶Lagrange系统和时间尺度联合Caputo分数阶Hamilton系统的Noether定理,并给出若干特例.有些特例的结果与原有结果一致,有些特例的结论是新的.广义分数阶导数是更为广泛的导数,目前时间尺度广义分数阶导数微积分尚未研究,后续也可考虑时间尺度广义分数阶Noether定理.

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