

变加速相对运动的广义高斯原理与动力学方程^{*}

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摘要 研究变加速相对运动的广义高斯原理与动力学方程. 首先, 通过对质点的相对运动动力学方程求导出力变率方程, 提出相对急动度空间的概念并在此空间构建变加速相对运动的广义高斯原理. 其次, 定义变加速相对运动的广义拘束函数, 证明真实运动使其取得极小. 引入相对运动的加速度能和急动度能, 导出广义坐标下变加速相对运动广义高斯原理的 Appell 形式、Lagrange 形式和 Nielsen 形式. 最后, 由所得原理导出完整和非完整系统变加速相对运动的动力学方程.

关键词 变加速动力学, 相对运动, 广义高斯原理, 非完整力学

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Generalized Gaussian Principle and Dynamic Equations of Relative Motion with Variable Acceleration^{*}

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Abstract The generalized Gaussian principle and dynamic equation of relative motion with variable acceleration are studied. Firstly, the equation of force change rate is obtained by derivation of the kinetic equation of relative motion of particle, and the concept of space of relative jerky is put forward and the generalized Gaussian principle of relative motion with variable acceleration is constructed in this space. Secondly, the generalized compulsion function of variable acceleration relative motion is defined, and it is proved that the real motion minimizes it. The Appell, Lagrange and Nielsen forms of the generalized Gaussian principle of relative motion with variable acceleration under generalized coordinates are derived by introducing the acceleration energy and the jerky energy of relative motion. Finally, the dynamics equations of the relative motion of holonomic and nonholonomic systems with variable acceleration are derived from the obtained principles.

Key words variable acceleration dynamics, relative motion, generalized Gaussian principle, non-holonomic mechanics

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引言

变分原理提供一种准则,依据这个准则可将真实运动从所有可能运动中挑选出来^[1].高斯原理是一个微分变分原理,在所有微分形式的变分原理中只有高斯原理是极值原理^[2-4],它可表示为拘束函数的高斯变分等于零.因此又可称高斯最小拘束原理.早在 20 世纪 80 年代,梅凤翔^[5]和刘恩远^[6]研究了约束依赖于质量变化或控制参数的高斯原理.黎邦隆和宋福磐^[7]研究了冲击问题的高斯原理.薛纭等^[8,9]将高斯原理推广到弹性杆动力学.刘延柱^[10,11]应用高斯原理于多体系统的动力学建模.姚文莉等^[12-14]应用高斯原理于非理想系统以及单侧约束系统的优化.最近,笔者将高斯最小拘束原理拓展到任意阶导数空间^[15],相对运动^[16]以及变加速运动^[17].变加速动力学涉及急动度(jerk)概念,亦称牛顿猝变动力学(Newtonian jerky dynamics)^[18-20].文献[21-23]介绍了变加速动力学,包括其历史背景与应用领域以及发展态势.变加速运动采用三阶微分方程描述,文献[24-26]研究了耗散系统,变质量力学,相对运动以及非独立坐标下的三阶微分方程.本文研究变加速相对运动的高斯原理和动力学方程.用分析力学的理论与方法研究力学系统的相对运动动力学,不仅可在表现形式上达到统一,而且对复杂系统显示出优越性^[27-31].文章安排如下:第 1 节提出相对急动度空间及其理想约束的概念,基于相对运动的动力学方程建立变加速相对运动的高斯原理.第 2 节,定义变加速相对运动的广义拘束函数,证明真实运动使得拘束函数取得极小值,从而建立变加速相对运动的高斯最小拘束原理,给出广义坐标下原理的三种形式.第 3 节,在所得原理基础上,建立完整和非完整约束系统的 Appell 方程, Lagrange 方程以及 Nielsen 方程.第 4 节为结论.

1 广义高斯原理

设力学系统由载体和被载体组成.前者视作刚体,其上固连动坐标系 $Ox'y'z'$;后者为 N 个质点,相对于动系的位形用广义坐标 $q_s (s = 1, 2, \dots, n)$ 描述.动系的角速度 $\boldsymbol{\omega}$ 以及极点 O 的速度 $\mathbf{v}_O$ 均为已知函数.第 i 个质点相对运动的动力学方程为^[28]

$$-m_i \ddot{\mathbf{r}}'_i + \mathbf{F}_i + \mathbf{N}_i + \mathbf{F}_{ci}^1 + \mathbf{F}_{ci}^1 = \mathbf{0} \quad (1)$$

其中 m_i 是质量; $\ddot{\mathbf{r}}'_i$ 是相对加速度,即动系中质点的矢径 $\mathbf{r}'_i = \mathbf{r}'_i(q_s, t)$ 对时间 t 的二阶相对导数; $\mathbf{F}_i$ 是主动力, $\mathbf{F}_{ci}^1 = -m_i \dot{\mathbf{v}}_O - m_i \dot{\boldsymbol{\omega}} \times \mathbf{r}'_i - m_i \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_i)$ 为牵连惯性力, $\mathbf{F}_{ci}^1 = -2m_i \boldsymbol{\omega} \times \dot{\mathbf{r}}'_i$ 为科氏惯性力, $\mathbf{N}_i$ 是约束反力, $i = 1, 2, \dots, N$.

将方程(1)对时间 t 求相对导数,得

$$-m_i \ddot{\mathbf{r}}'_i + \ddot{\mathbf{F}}_i + \ddot{\mathbf{N}}_i + \ddot{\mathbf{F}}_{ci}^1 + \ddot{\mathbf{F}}_{ci}^1 = \mathbf{0} \quad (2)$$

式(2)称为相对力变率方程.将其点乘 $\delta_G^{(1)} \dot{\mathbf{r}}'_i$ 并对 i 求和,得

$$\sum_{i=1}^N (-m_i \ddot{\mathbf{r}}'_i + \ddot{\mathbf{F}}_i + \ddot{\mathbf{N}}_i + \ddot{\mathbf{F}}_{ci}^1 + \ddot{\mathbf{F}}_{ci}^1) \cdot \delta_G^{(1)} \dot{\mathbf{r}}'_i = 0 \quad (3)$$

其中 $\delta_G^{(1)}(\cdot)$ 表示相对加速度的导数空间(或称相对急动度空间)中高斯意义下的变分,或称广义高斯变分,有

$$\delta_G^{(1)} t = 0, \delta_G^{(1)} \mathbf{r}'_i = \delta_G^{(1)} \dot{\mathbf{r}}'_i = \delta_G^{(1)} \ddot{\mathbf{r}}'_i = \mathbf{0},$$

$$\delta_G^{(1)} \ddot{\mathbf{r}}'_i \neq \mathbf{0} \quad (4)$$

在相对急动度空间,高斯意义下的理想约束条件为

$$\sum_{i=1}^N \ddot{\mathbf{N}}_i \cdot \delta_G^{(1)} \ddot{\mathbf{r}}'_i \neq 0 \quad (5)$$

将式(5)代入式(3),得

$$\sum_{i=1}^N (-m_i \ddot{\mathbf{r}}'_i + \ddot{\mathbf{F}}_i + \ddot{\mathbf{F}}_{ci}^1 + \ddot{\mathbf{F}}_{ci}^1) \cdot \delta_G^{(1)} \ddot{\mathbf{r}}'_i \neq 0 \quad (6)$$

式(6)称为变加速相对运动动力学的广义高斯原理.

2 广义高斯最小拘束原理

定义变加速相对运动的广义拘束函数为

$$Z_r^{(1)} = \sum_{i=1}^N \frac{1}{2} m_i \left(\ddot{\mathbf{r}}'_i - \frac{\ddot{\mathbf{F}}_i + \ddot{\mathbf{F}}_{ci}^1 + \ddot{\mathbf{F}}_{ci}^1}{m_i} \right)^2 \quad (7)$$

则原理(6)成为

$$\delta_G^{(1)} Z_r^{(1)} = 0 \quad (8)$$

实际上,对式(7)求广义高斯变分,有

$$\begin{aligned} \delta_G^{(1)} Z_r^{(1)} &= \sum_{i=1}^N \left[m_i \left(\ddot{\mathbf{r}}'_i - \frac{\ddot{\mathbf{F}}_i + \ddot{\mathbf{F}}_{ci}^1 + \ddot{\mathbf{F}}_{ci}^1}{m_i} \right) \times \right. \\ &\quad \left. \delta_G^{(1)} \left(\ddot{\mathbf{r}}'_i - \frac{\ddot{\mathbf{F}}_i + \ddot{\mathbf{F}}_{ci}^1 + \ddot{\mathbf{F}}_{ci}^1}{m_i} \right) \right] \\ &= \sum_{i=1}^N m_i \left(\ddot{\mathbf{r}}'_i - \frac{\ddot{\mathbf{F}}_i + \ddot{\mathbf{F}}_{ci}^1 + \ddot{\mathbf{F}}_{ci}^1}{m_i} \right) \cdot \delta_G^{(1)} \ddot{\mathbf{r}}'_i \end{aligned} \quad (9)$$

将式(6)代入式(9),即得式(8).

假设 $\tilde{\mathbf{r}}'_{(3)i}$ 为真实运动中质点的相对急动度, $\tilde{\mathbf{r}}'_{(3)i} + \delta_G^{(1)} \tilde{\mathbf{r}}'_{(3)i}$ 是可能运动的相对急动度,两者相应的拘束函数之差为

$$\begin{aligned} \Delta Z_r^{(1)} &= \sum_{i=1}^N \frac{1}{2} m_i \left[\left(\tilde{\mathbf{r}}'_{(3)i} + \delta_G^{(1)} \tilde{\mathbf{r}}'_{(3)i} - \frac{\tilde{\mathbf{F}}_i + \tilde{\mathbf{F}}_{ei}^1 + \tilde{\mathbf{F}}_{ci}^1}{m_i} \right)^2 - \left(\tilde{\mathbf{r}}'_{(3)i} - \frac{\tilde{\mathbf{F}}_i + \tilde{\mathbf{F}}_{ei}^1 + \tilde{\mathbf{F}}_{ci}^1}{m_i} \right)^2 \right] = \sum_{i=1}^N \frac{1}{2} m_i (\delta_G^{(1)} \tilde{\mathbf{r}}'_{(3)i})^2 + \\ &\sum_{i=1}^N m_i \left(\tilde{\mathbf{r}}'_{(3)i} - \frac{\tilde{\mathbf{F}}_i + \tilde{\mathbf{F}}_{ei}^1 + \tilde{\mathbf{F}}_{ci}^1}{m_i} \right) \cdot \delta_G^{(1)} \tilde{\mathbf{r}}'_{(3)i} \\ &= \sum_{i=1}^N \frac{1}{2} m_i (\delta_G^{(1)} \tilde{\mathbf{r}}'_{(3)i})^2 > 0 \end{aligned} \quad (10)$$

从式(10)可见,对真实运动而言,其相对急动度使拘束函数 $Z_r^{(1)}$ 极小.

式(8)可称为变加速相对运动的广义高斯最小拘束原理.

展开式(7),得

$$Z_r^{(1)} = \sum_{i=1}^N \frac{1}{2} m_i \tilde{\mathbf{r}}'_{(3)i} \cdot \tilde{\mathbf{r}}'_{(3)i} - \sum_{i=1}^N (\tilde{\mathbf{F}}_i + \tilde{\mathbf{F}}_{ei}^1 + \tilde{\mathbf{F}}_{ci}^1) \cdot \tilde{\mathbf{r}}'_{(3)i} + \dots \quad (11)$$

这里“...”表示与相对急动度无关的项.

引进相对运动的加速度能量 S_r 和相对急动度能量 $S_r^{(1)}$,即有

$$S_r = \frac{1}{2} \sum_{i=1}^N m_i \tilde{\mathbf{r}}'_{(3)i} \cdot \tilde{\mathbf{r}}'_{(3)i} \quad (12)$$

和

$$S_r^{(1)} = \frac{1}{2} \sum_{i=1}^N m_i \tilde{\mathbf{r}}'_{(3)i} \cdot \tilde{\mathbf{r}}'_{(3)i} \quad (13)$$

将矢径 $\mathbf{r}'_i = \mathbf{r}'_i(q_s, t)$ 对时间求 m 阶相对导数,得到

$$\tilde{\mathbf{r}}'_{(3)i} = \sum_{s=1}^n \frac{\partial \mathbf{r}'_i}{\partial q_s} \dot{q}_s + \frac{\partial \mathbf{r}'_i}{\partial t} \quad (14)$$

$$\begin{aligned} \tilde{\mathbf{r}}'_{(3)i} &= \sum_{s=1}^n \frac{\partial \mathbf{r}'_i}{\partial q_s} \ddot{q}_s + \sum_{s=1}^n \sum_{k=1}^n \frac{\partial^2 \mathbf{r}'_i}{\partial q_s \partial q_k} \dot{q}_k \dot{q}_s + \\ &2 \sum_{s=1}^n \frac{\partial^2 \mathbf{r}'_i}{\partial q_s \partial t} \dot{q}_s + \frac{\partial^2 \mathbf{r}'_i}{\partial t^2} \end{aligned} \quad (15)$$

$$\begin{aligned} \tilde{\mathbf{r}}'_{(3)i} &= \sum_{s=1}^n \frac{\partial \mathbf{r}'_i}{\partial q_s} \ddot{q}_s + 3 \sum_{s=1}^n \left(\sum_{k=1}^n \frac{\partial^2 \mathbf{r}'_i}{\partial q_s \partial q_k} \dot{q}_k + \frac{\partial^2 \mathbf{r}'_i}{\partial q_s \partial t} \right) \dot{q}_s + \\ &\sum_{s=1}^n \sum_{k=1}^n \sum_{j=1}^n \frac{\partial^3 \mathbf{r}'_i}{\partial q_s \partial q_k \partial q_j} \dot{q}_s \dot{q}_k \dot{q}_j + \\ &3 \sum_{s=1}^n \left(\sum_{k=1}^n \frac{\partial^3 \mathbf{r}'_i}{\partial q_s \partial q_k \partial t} \dot{q}_k + \frac{\partial^3 \mathbf{r}'_i}{\partial q_s \partial t^2} \right) \dot{q}_s + \frac{\partial^3 \mathbf{r}'_i}{\partial t^3} \end{aligned} \quad (16)$$

由式(16),得

$$\delta_G^{(1)} \tilde{\mathbf{r}}'_{(3)i} = \sum_{s=1}^n \frac{\partial \mathbf{r}'_i}{\partial q_s} \delta_G^{(1)} \dot{q}_s \quad (17)$$

对式(11)求广义高斯变分,并利用式(17),得

$$\begin{aligned} \delta_G^{(1)} Z_i^{(1)} &= \sum_{s=1}^n \sum_{i=1}^N m_i \tilde{\mathbf{r}}'_{(3)i} \cdot \frac{\partial \mathbf{r}'_i}{\partial q_s} \delta_G^{(1)} \dot{q}_s - \\ &\sum_{s=1}^n \sum_{i=1}^N (\tilde{\mathbf{F}}_i + \tilde{\mathbf{F}}_{ei}^1 + \tilde{\mathbf{F}}_{ci}^1) \cdot \frac{\partial \mathbf{r}'_i}{\partial q_s} \delta_G^{(1)} \dot{q}_s \end{aligned} \quad (18)$$

由式(14)~(16)易知

$$\frac{\partial \tilde{\mathbf{r}}'_{(3)i}}{\partial \dot{q}_s} = \frac{\partial \tilde{\mathbf{r}}'_{(3)i}}{\partial \ddot{q}_s} = \frac{\partial \tilde{\mathbf{r}}'_{(3)i}}{\partial \dot{q}_s} = \frac{\partial \mathbf{r}'_i}{\partial q_s} \quad (19)$$

$$\frac{\partial \tilde{\mathbf{r}}'_{(3)i}}{\partial \dot{q}_s} = 2 \frac{\partial \tilde{\mathbf{r}}'_{(3)i}}{\partial q_s} = 2 \frac{d}{dt} \frac{\partial \mathbf{r}'_i}{\partial q_s} \quad (20)$$

利用式(19)和(13),得

$$\begin{aligned} \sum_{i=1}^N m_i \tilde{\mathbf{r}}'_{(3)i} \cdot \frac{\partial \mathbf{r}'_i}{\partial q_s} &= \frac{\partial}{\partial \dot{q}_s} \left(\sum_{i=1}^N \frac{1}{2} m_i \tilde{\mathbf{r}}'_{(3)i} \cdot \tilde{\mathbf{r}}'_{(3)i} \right) \\ &= \frac{\partial S_r^{(1)}}{\partial \dot{q}_s} \end{aligned} \quad (21)$$

式(18)等号右边第二项中

$$Q_s^{(1)} = \sum_{i=1}^N \tilde{\mathbf{F}}_i \cdot \frac{\partial \mathbf{r}'_i}{\partial q_s} \quad (22)$$

称为广义主动力变率^[26],而

$$\begin{aligned} \sum_{i=1}^N (\tilde{\mathbf{F}}_{ei}^1 + \tilde{\mathbf{F}}_{ci}^1) \cdot \frac{\partial \mathbf{r}'_i}{\partial q_s} &= - \sum_{i=1}^N \frac{d}{dt} [m_i \dot{\mathbf{v}}_O + m_i \dot{\boldsymbol{\omega}} \times \\ &\mathbf{r}'_i + m_i \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_i) + 2m_i \boldsymbol{\omega} \times \tilde{\mathbf{r}}'_{(3)i}] \cdot \frac{\partial \mathbf{r}'_i}{\partial q_s} \end{aligned} \quad (23)$$

下面我们分析式(23)右端各项. 第一项为

$$\begin{aligned} \sum_{i=1}^N \frac{d}{dt} (m_i \dot{\mathbf{v}}_O) \cdot \frac{\partial \mathbf{r}'_i}{\partial q_s} &= \frac{\partial}{\partial \dot{q}_s} \left[\frac{d}{dt} (\dot{\mathbf{v}}_O + \boldsymbol{\omega} \times \mathbf{v}_O) \cdot \sum_{i=1}^N m_i \tilde{\mathbf{r}}'_{(3)i} \right] \\ &= \frac{\partial}{\partial \dot{q}_s} \left[\frac{d}{dt} (\dot{\mathbf{v}}_O + \boldsymbol{\omega} \times \mathbf{v}_O) \cdot M \tilde{\mathbf{r}}'_{(3)c} \right] = \frac{\partial}{\partial \dot{q}_s} \Pi^{O(1)} \end{aligned} \quad (24)$$

其中

$$\Pi^{O(1)} = \frac{d}{dt} (\dot{\mathbf{v}}_O + \boldsymbol{\omega} \times \mathbf{v}_O) \cdot M \tilde{\mathbf{r}}'_{(3)c} \quad (25)$$

可称为质点系的平动惯性力变率的势函数. 由于 $(-m_i \boldsymbol{\omega} \times \tilde{\mathbf{r}}'_{(3)i})$ 是质点的旋转惯性力, 因此第二项为

$$- \sum_{i=1}^N \frac{d}{dt} (m_i \dot{\boldsymbol{\omega}} \times \mathbf{r}'_i) \cdot \frac{\partial \mathbf{r}'_i}{\partial q_s} = Q_s^{\dot{\boldsymbol{\omega}}(1)} \quad (26)$$

可称为广义旋转惯性力变率,第三项为

$$\sum_{i=1}^N \frac{d}{dt} [m_i \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_i)] \cdot \frac{\partial \mathbf{r}'_i}{\partial \dot{q}_s} = \frac{\partial}{\partial \dot{q}_s} \Pi^{\omega(1)} \quad (27)$$

其中

$$\begin{aligned} \Pi^{\omega(1)} = & - \sum_{i=1}^N m_i \left[\frac{1}{2} (\boldsymbol{\omega} \times \tilde{\mathbf{r}}'_i) \cdot (\boldsymbol{\omega} \times \tilde{\mathbf{r}}'_i) + \right. \\ & \left. (\boldsymbol{\omega} \times \mathbf{r}'_i) \cdot (\tilde{\boldsymbol{\omega}} \times \tilde{\mathbf{r}}'_i) + (\tilde{\boldsymbol{\omega}} \times \mathbf{r}'_i) \cdot (\boldsymbol{\omega} \times \tilde{\mathbf{r}}'_i) \right] \end{aligned} \quad (28)$$

称为转动惯性力变率的势函数,第四项为

$$\begin{aligned} & - \sum_{i=1}^N \frac{d}{dt} (2m_i \boldsymbol{\omega} \times \tilde{\mathbf{r}}'_i) \cdot \frac{\partial \mathbf{r}'_i}{\partial \dot{q}_s} \\ & = \sum_{j=0}^1 \Gamma_s^{1-j} = \sum_{j=0}^1 \sum_{k=1}^n \gamma_{sk}^{j(2-j)} \dot{q}_k \end{aligned} \quad (29)$$

称为广义陀螺力变率,其中

$$\gamma_{sk}^j = 2\tilde{\boldsymbol{\omega}} \cdot \sum_{i=1}^N m_i \frac{\partial \mathbf{r}'_i}{\partial \dot{q}_s} \times \frac{\partial \mathbf{r}'_i}{\partial \dot{q}_k} = -\gamma_{ks}^j \quad (30)$$

将式(21)、(22)、(24)、(26)、(27)和(29)代入式(18),得

$$\begin{aligned} \delta_G^{(1)} Z_r^{(1)} = & \sum_{s=1}^n \left\{ \frac{\partial S_r^{(1)}}{\partial \dot{q}_s^{(3)}} - Q_s^{(1)} + \right. \\ & \left. \frac{\partial}{\partial \dot{q}_s} (\Pi^{o(1)} + \Pi^{\omega(1)}) - Q_s^{\dot{\omega}(1)} - \sum_{j=0}^1 \Gamma_s^{1-j} \right\} \delta_G^{(1)(3)} \dot{q}_s \end{aligned} \quad (31)$$

于是,原理(8)成为

$$\begin{aligned} & \sum_{s=1}^n \left\{ \frac{\partial S_r^{(1)}}{\partial \dot{q}_s^{(3)}} - Q_s^{(1)} + \right. \\ & \left. \frac{\partial}{\partial \dot{q}_s} (\Pi^{o(1)} + \Pi^{\omega(1)}) - Q_s^{\dot{\omega}(1)} - \sum_{j=0}^1 \Gamma_s^{1-j} \right\} \delta_G^{(1)(3)} \dot{q}_s = 0 \end{aligned} \quad (32)$$

式(32)称为广义坐标下变加速相对运动动力学的高斯原理的 Appell 形式.

如果取 $\mathbf{v}_O = \boldsymbol{\omega} = 0$, 则

$$\Pi^{o(1)} + \Pi^{\omega(1)} = Q_s^{\dot{\omega}(1)} = \Gamma_s^{1-j} = 0 \quad (33)$$

于是原理(32)成为

$$\sum_{s=1}^n \left\{ \frac{\partial S_r^{(1)}}{\partial \dot{q}_s^{(3)}} - Q_s^{(1)} \right\} \delta_G^{(1)(3)} \dot{q}_s = 0 \quad (34)$$

这是变加速绝对运动的广义高斯原理的 Appell 形式^[17],其中 $S^{(1)} = \frac{1}{2} \sum_{i=1}^N m_i \dot{\mathbf{r}}_i^{(3)} \cdot \dot{\mathbf{r}}_i^{(3)}$ 是绝对运动的急动度能量.

利用式(19)和(20),容易证明:

$$\sum_{i=1}^N m_i \tilde{\mathbf{r}}'_i \cdot \frac{\partial \mathbf{r}'_i}{\partial \dot{q}_s} = \frac{d}{dt} \left[\sum_{i=1}^N \frac{1}{2} m_i \tilde{\mathbf{r}}'_i \cdot \tilde{\mathbf{r}}'_i \right] -$$

$$\frac{1}{2} \frac{\partial}{\partial \dot{q}_s} \left(\sum_{i=1}^N \frac{1}{2} m_i \tilde{\mathbf{r}}'_i \cdot \tilde{\mathbf{r}}'_i \right) = \frac{d}{dt} \frac{\partial S_r}{\partial \dot{q}_s} - \frac{1}{2} \frac{\partial S_r}{\partial \dot{q}_s} \quad (35)$$

$$\sum_{i=1}^N m_i \tilde{\mathbf{r}}'_i \cdot \frac{\partial \mathbf{r}'_i}{\partial \dot{q}_s} = \frac{\partial}{\partial \dot{q}_s} \frac{dS_r}{dt} - \frac{3}{2} \frac{\partial S_r}{\partial \dot{q}_s} \quad (36)$$

由式(21)、(35)和(36),原理(32)可分别表示为

$$\begin{aligned} & \sum_{s=1}^n \left\{ \frac{d}{dt} \frac{\partial S_r}{\partial \dot{q}_s} - \frac{1}{2} \frac{\partial S_r}{\partial \dot{q}_s} - Q_s^{(1)} + \right. \\ & \left. \frac{\partial}{\partial \dot{q}_s} (\Pi^{o(1)} + \Pi^{\omega(1)}) - Q_s^{\dot{\omega}(1)} - \sum_{j=0}^1 \Gamma_s^{1-j} \right\} \delta_G^{(1)(3)} \dot{q}_s \\ & = 0 \end{aligned} \quad (37)$$

以及

$$\begin{aligned} & \sum_{s=1}^n \left\{ \frac{\partial}{\partial \dot{q}_s} \frac{dS_r}{dt} - \frac{3}{2} \frac{\partial S_r}{\partial \dot{q}_s} - Q_s^{(1)} + \right. \\ & \left. \frac{\partial}{\partial \dot{q}_s} (\Pi^{o(1)} + \Pi^{\omega(1)}) - Q_s^{\dot{\omega}(1)} - \sum_{j=0}^1 \Gamma_s^{1-j} \right\} \delta_G^{(1)(3)} \dot{q}_s \\ & = 0 \end{aligned} \quad (38)$$

式(37)和(38)分别称为广义坐标下变加速相对运动动力学的高斯原理的 Lagrange 形式和 Nielsen 形式.

3 变加速动力学方程

对于完整系统, $\delta_G^{(1)(3)} \dot{q}_s$ 是彼此独立且任意的, 因此由原理(32)得到

$$\begin{aligned} & \frac{\partial S_r^{(1)}}{\partial \dot{q}_s^{(3)}} - Q_s^{(1)} + \frac{\partial}{\partial \dot{q}_s} (\Pi^{o(1)} + \Pi^{\omega(1)}) - \\ & Q_s^{\dot{\omega}(1)} - \sum_{j=0}^1 \Gamma_s^{1-j} = 0 \end{aligned} \quad (39)$$

式(39)是变加速相对运动的 Appell 方程.

由原理(37),得到

$$\begin{aligned} & \frac{d}{dt} \frac{\partial S_r}{\partial \dot{q}_s} - \frac{1}{2} \frac{\partial S_r}{\partial \dot{q}_s} - Q_s^{(1)} + \frac{\partial}{\partial \dot{q}_s} (\Pi^{o(1)} + \Pi^{\omega(1)}) - \\ & Q_s^{\dot{\omega}(1)} - \sum_{j=0}^1 \Gamma_s^{1-j} = 0 \end{aligned} \quad (40)$$

式(40)是变加速相对运动的 Lagrange 方程.

由原理(38),得到

$$\begin{aligned} & \frac{\partial}{\partial \dot{q}_s} \frac{dS_r}{dt} - \frac{3}{2} \frac{\partial S_r}{\partial \dot{q}_s} - Q_s^{(1)} + \frac{\partial}{\partial \dot{q}_s} (\Pi^{o(1)} + \Pi^{\omega(1)}) - \\ & Q_s^{\dot{\omega}(1)} - \sum_{j=0}^1 \Gamma_s^{1-j} = 0 \end{aligned} \quad (41)$$

式(41)是变加速相对运动的 Nielsen 方程.

如果系统受到理想三阶非线性非完整约束

$$\phi_a = \phi_a(t, q_s, \dot{q}_s, \ddot{q}_s, \dot{q}_s^{(3)}) = 0 \quad (42)$$

其中 $\alpha=1,2,\cdots,g$. 在相对急动度空间,虚位移满足

$$\sum_{s=1}^n \frac{\partial \phi_a}{\partial \dot{q}_s} \delta_G^{(1)} q_s = 0 \quad (43)$$

由原理(32)和式(43),采用 Lagrange 乘子法,得到

$$\begin{aligned} & \frac{\partial S_r^{(1)}}{\partial \dot{q}_s^{(3)}} - Q_s^{(1)} + \frac{\partial}{\partial \dot{q}_s} (\Pi^{o(1)} + \Pi^{\omega(1)}) - \\ & Q_s^{\dot{\omega}(1)} - \sum_{j=0}^1 \Gamma_s^{1-j} + \sum_{a=1}^g \mu_a \frac{\partial \phi_a}{\partial \dot{q}_s^{(3)}} = 0 \end{aligned} \quad (44)$$

式(44)是三阶非完整系统变加速相对运动的 Appell 方程.

类似地,由原理(37)和式(43),可得到

$$\begin{aligned} & \frac{d}{dt} \frac{\partial S_r}{\partial \ddot{q}_s} - \frac{1}{2} \frac{\partial S_r}{\partial \dot{q}_s} - Q_s^{(1)} + \frac{\partial}{\partial \dot{q}_s} (\Pi^{o(1)} + \Pi^{\omega(1)}) - \\ & Q_s^{\dot{\omega}(1)} - \sum_{j=0}^1 \Gamma_s^{1-j} + \sum_{a=1}^g \mu_a \frac{\partial \phi_a}{\partial \dot{q}_s^{(3)}} = 0 \end{aligned} \quad (45)$$

式(45)是三阶非完整系统变加速相对运动的 Lagrange 方程.

由原理(38)和式(43),可得到

$$\begin{aligned} & \frac{\partial}{\partial \ddot{q}_s} \frac{dS_r}{dt} - \frac{3}{2} \frac{\partial S_r}{\partial \dot{q}_s} - Q_s^{(1)} + \frac{\partial}{\partial \dot{q}_s} (\Pi^{o(1)} + \Pi^{\omega(1)}) - \\ & Q_s^{\dot{\omega}(1)} - \sum_{j=0}^1 \Gamma_s^{1-j} + \sum_{a=1}^g \mu_a \frac{\partial \phi_a}{\partial \dot{q}_s^{(3)}} = 0 \end{aligned} \quad (46)$$

式(46)是三阶非完整系统变加速相对运动的 Nielsen 方程.

如果受理想一阶非完整约束

$$\phi_a = \phi_a(t, q_s, \dot{q}_s) = 0 \quad (47)$$

则可将方程(47)求二阶导数,得

$$\ddot{\phi}_a = \sum_{s=1}^n \frac{\partial \phi_a}{\partial \dot{q}_s} \ddot{q}_s + \cdots = 0 \quad (48)$$

于是虚位移满足

$$\sum_{s=1}^n \frac{\partial \phi_a}{\partial \dot{q}_s} \delta_G^{(1)} q_s = 0 \quad (49)$$

则利用式(49),结合原理(32)、(37)和(38),可得到一阶非完整系统变加速相对运动的 Appell 方程

$$\begin{aligned} & \frac{\partial S_r^{(1)}}{\partial \dot{q}_s^{(3)}} - Q_s^{(1)} + \frac{\partial}{\partial \dot{q}_s} (\Pi^{o(1)} + \Pi^{\omega(1)}) - \\ & Q_s^{\dot{\omega}(1)} - \sum_{j=0}^1 \Gamma_s^{1-j} + \sum_{a=1}^g \mu_a \frac{\partial \phi_a}{\partial \dot{q}_s^{(3)}} = 0 \end{aligned} \quad (50)$$

Lagrange 方程

$$\begin{aligned} & \frac{d}{dt} \frac{\partial S_r}{\partial \ddot{q}_s} - \frac{1}{2} \frac{\partial S_r}{\partial \dot{q}_s} - Q_s^{(1)} + \frac{\partial}{\partial \dot{q}_s} (\Pi^{o(1)} + \Pi^{\omega(1)}) - \\ & Q_s^{\dot{\omega}(1)} - \sum_{j=0}^1 \Gamma_s^{1-j} + \sum_{a=1}^g \mu_a \frac{\partial \phi_a}{\partial \dot{q}_s^{(3)}} = 0 \end{aligned} \quad (51)$$

以及 Nielsen 方程

$$\begin{aligned} & \frac{\partial}{\partial \ddot{q}_s} \frac{dS_r}{dt} - \frac{3}{2} \frac{\partial S_r}{\partial \dot{q}_s} - Q_s^{(1)} + \frac{\partial}{\partial \dot{q}_s} (\Pi^{o(1)} + \Pi^{\omega(1)}) - \\ & Q_s^{\dot{\omega}(1)} - \sum_{j=0}^1 \Gamma_s^{1-j} + \sum_{a=1}^g \mu_a \frac{\partial \phi_a}{\partial \dot{q}_s^{(3)}} = 0 \end{aligned} \quad (52)$$

其中 μ_a 为约束乘子.

如果约束是二阶的,可作类似分析,此处不再赘述.

4 结论

变加速动力学由于其不仅与非线性动力学以及混沌现象密切相关,而且在工程技术、日常生活中都有实际应用,如结构物抗风抗震、车辆乘坐的舒适度等,同时相对运动处处存在,因此研究变加速相对运动的高斯原理具有重要意义.文章在文献[17]的基础上,进一步提出了相对急动度空间的概念以及理想约束的条件,建立变加速相对运动的广义高斯原理.定义广义拘束函数,证明变加速相对运动的广义高斯最小拘束原理.引入相对运动的急动度能量,导出广义坐标下变加速相对运动的高斯原理和动力学方程.

参考文献

- [1] 梅凤翔, 吴惠彬, 李彦敏. 分析力学史略[M]. 北京: 科学出版社, 2019.
MEI F X, WU H B, LI Y M. A brief history of analytical mechanics [M]. Beijing: Science Press, 2019. (in Chinese)
- [2] 梅凤翔, 李彦敏, 吴惠彬. 关于 Gauss 原理[J]. 动力学与控制学报, 2016, 14(4): 301-306.
MEI F X, LI Y M, WU H B. On the Gauss principle [J]. Journal of Dynamics and Control, 2016, 14(4): 301-306. (in Chinese)
- [3] 陈滨. 分析动力学[M]. 2版. 北京: 北京大学出版社, 2012.
CHEN B. Analytical dynamics [M]. 2nd ed. Beijing: Peking University Press, 2012. (in Chinese)
- [4] 杰格日达, 索尔塔哈诺夫, 尤士科夫. 非完整系统的运动方程和力学的变分原理: 新一类控制问题[M]. 梅凤翔译. 北京: 北京理工大学出版社, 2007.
ZEGZHDA S A, SOLTAKHANOV S K, YUSKOV M P. Equations of motion for nonholonomic

- systems and variational principles of mechanics: a new class of control problems [M]. Beijing: Beijing Institute of Technology Press, 2007. (in Chinese)
- [5] 梅凤翔. 约束依赖于质量变化过程的力学系统的 Appell 方程[J]. 北京工业学院学报, 1986, 6(4): 19—29.
- MEI F X. Appell equations for mechanical systems with constraints depending on the process of mass variation [J]. Journal of Beijing Institute of Technology, 1986, 6(4): 19—29. (in Chinese)
- [6] 刘恩远. 变质量可控力学系统的 Gauss 原理和 Appell 方程[J]. 固体力学学报, 1986, 7(2): 122—129.
- LIU E Y. Gauss principle and Appell equation of dynamic control system with variable mass [J]. Acta Mechanica Solida Sinica, 1986, 7(2): 122—129. (in Chinese)
- [7] 黎邦隆, 宋福磐. 关于冲力情况下的高斯最小约束原理[J]. 湖南大学学报(自然科学版), 1995, 22(4): 23—28.
- LI B L, SONG F P. On Gauss' principle of least constraint for impulsive motion [J]. Journal of Hunan University (Natural Sciences), 1995, 22(4): 23—28. (in Chinese)
- [8] 薛纭, 翁德玮. 超细长弹性杆动力学的 Gauss 原理[J]. 物理学报, 2009, 58(1): 34—39.
- XUE Y, WENG D W. Gauss principle for a super-thin elastic rod dynamics [J]. Acta Physica Sinica, 2009, 58(1): 34—39. (in Chinese)
- [9] 薛纭, 曲佳乐, 陈立群. Cosserat 生长弹性杆动力学的 Gauss 最小约束原理[J]. 应用数学和力学, 2015, 36(7): 700—709.
- XUE Y, QU J L, CHEN L Q. Gauss principle of least constraint for Cosserat growing elastic rod dynamics [J]. Applied Mathematics and Mechanics, 2015, 36(7): 700—709. (in Chinese)
- [10] 刘延柱. 基于高斯原理的多体系统动力学建模[J]. 力学学报, 2014, 46(6): 940—945.
- LIU Y Z. Dynamic modeling of multi-body system based on Gauss's principle [J]. Chinese Journal of Theoretical and Applied Mechanics, 2014, 46(6): 940—945. (in Chinese)
- [11] 刘延柱. 杆网系统基于高斯原理的动力学建模[J]. 动力学与控制学报, 2018, 16(4): 289—294.
- LIU Y Z. Dynamical modeling of a net system of rods based on Gauss's principle [J]. Journal of Dynamics and Control, 2018, 16(4): 289—294. (in Chinese)
- [12] 姚文莉, 刘彦平, 杨流松. 基于高斯原理的非理想系统动力学建模[J]. 力学学报, 2020, 52(4): 945—953.
- YAO W L, LIU Y P, YANG L S. Dynamic modeling of nonideal system based on Gauss's principle [J]. Chinese Journal of Theoretical and Applied Mechanics, 2020, 52(4): 945—953. (in Chinese)
- [13] YAO W L, YANG L S, GUO M M. Gauss optimization method for the dynamics of unilateral contact of rigid multibody systems [J]. Acta Mechanica Sinica, 2021, 37(3): 494—506.
- [14] YAO W L, YANG L S, SONG K W, et al. Optimization method for dynamics of non-holonomic system based on Gauss' principle [J]. Acta Mechanica Sinica, 2020, 36(5): 1133—1141.
- [15] 张毅, 陈欣雨. 变质量力学系统的广义高斯原理及其对高阶非完整系统的推广[J]. 力学学报, 2022, 54(10): 2883—2891.
- ZHANG Y, CHEN X Y. The generalized Gauss principle for mechanical system with variable mass and its generalization to higher order nonholonomic systems [J]. Chinese Journal of Theoretical and Applied Mechanics, 2022, 54(10): 2883—2891. (in Chinese)
- [16] ZHANG Y, XIA J L. Gauss principle of least compulsion for relative motion dynamics and differential equations of motion [J]. Wuhan University Journal of Natural Sciences, 2024, 29(3): 273—283.
- [17] 张毅, 宋传静, 翟相华. 变加速动力学系统的广义高斯最小约束原理[J]. 力学学报, 2023, 55(5): 1174—1180.
- ZHANG Y, SONG C J, ZHAI X H. Generalized Gauss principle of least compulsion for variable acceleration dynamical systems [J]. Chinese Journal of Theoretical and Applied Mechanics, 2023, 55(5): 1174—1180. (in Chinese)
- [18] SCHOT S H. Jerk; The time rate of change of acceleration [J]. American Journal of Physics, 1978, 46(11): 1090—1094.
- [19] LINZ S J. Newtonian jerky dynamics: Some general properties [J]. American Journal of Physics, 1998, 66(12): 1109—1114.
- [20] LINZ S J. Nonlinear dynamical models and jerky motion [J]. American Journal of Physics, 1997, 65(6): 523—526.
- [21] 黄沛天, 马善钧, 徐学翔, 等. 变加速动力学纵横[J]. 力学与实践, 2004, 26(6): 85—87.

HUANG P T, MA S J, XU X X, et al. A review of the dynamics of varying accelerated motion [J]. *Mechanics in Engineering*, 2004, 26(6): 85—87. (in Chinese)

[22] 黄沛天, 黄文, 胡利云. 关于变加速动力学及其应用[J]. *力学与实践*, 2004, 26(1): 67—68.

HUANG P T, HUANG W, HU L Y. On the dynamics of varying acceleration and its application [J]. *Mechanics in Engineering*, 2004, 26(1): 67—68. (in Chinese)

[23] 余守宪, 董水金. 变加速动力学与舒适性问题[J]. *物理与工程*, 2006, 16(6): 35—37.

SHE S X, DONG S J. Varying accelerated motion and comfort [J]. *Physics and Engineering*, 2006, 16(6): 35—37. (in Chinese)

[24] MA S J, LIU M P, HUANG P T. The forms of three-order Lagrangian equation in relative motion [J]. *Chinese Physics*, 2005, 14(2): 244—246.

[25] MA S J, GE W G, HUANG P T. The third-order Lagrange equation for mechanical systems of variable mass [J]. *Chinese Physics*, 2005, 14(5): 879—881.

[26] 马善钧, 徐学翔, 黄沛天, 等. 完整系统三阶 Lagrange 方程的一种推导与讨论 [J]. *物理学报*, 2004, 53(11): 3648—3651.

MA S J, XU X X, HUANG P T, et al. The discussion on Lagrange equation containing third order derivatives [J]. *Acta Physica Sinica*, 2004, 53(11): 3648—3651. (in Chinese)

[27] LUR'E A I. *Analytical Mechanics* [M]. Moscow: GIFML, 1961.

[28] 梅凤翔. 分析力学专题 [M]. 北京: 北京工业学院出版社, 1988.

MEI F X. *Analytical mechanical special problem* [M]. Beijing: Beijing Institute of Technology Press, 1988. (in Chinese)

[29] GONG S N, GAO H F, FU J L. Lie symmetries of the relative motion systems on time scales [J]. *Indian Journal of Physics*, 2020, 94(3): 371—377.

[30] 梅凤翔, 吴惠彬. 相对运动动力学系统的 Lagrange 对称性[J]. *物理学报*, 2009, 58(9): 5919—5922.

MEI F X, WU H B. Lagrange symmetry for a dynamical system of relative motion [J]. *Acta Physica Sinica*, 2009, 58(9): 5919—5922. (in Chinese)

[31] ZHANG Y, SHANG M. Poisson theory and integration method for a dynamical system of relative motion [J]. *Chinese Physics B*, 2011, 20(2): 024501.