

一种几何精确梁的 Poisson 积分子^{*}

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摘要 在 Hamilton 力学的 Hamel 形式下, 针对无穷维力学系统的模拟问题提出了一种快速的几何数值积分算法. 首先, 引入对偶标架算子, 并借此导出约化 Poisson 括号, 所得 Hamilton 方程组能够复原 Hamel 场方程及其相容性条件. 通过离散 Poisson 括号结合辛 Euler 格式和隐式中点格式得到 Poisson 积分子. 其次, 以几何精确梁的运动学模型为例, 推导连续和离散情形下的约化 Poisson 括号和 Hamilton 方程组, 获得几何精确梁的 Poisson 积分子. 最后, 通过数值仿真验证了该 Poisson 积分子在保持能量和动量的同时, 相较于 Hamel 场积分子提升了计算效率.

关键词 Hamel 形式, Poisson 积分子, 几何数值算法

中图分类号: O302; O33; 3242; O31

文献标志码: A

A Poisson Integrator for Geometric Exact Beam^{*}

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Abstract Under the Hamel's formulism of Hamiltonian mechanics, a fast discrete geometric numerical integration algorithm is proposed for the simulation of infinite-dimensional mechanical systems. First, a dual frame operator is introduced, based on which the reduced Poisson bracket is derived. The resulting Hamiltonian equations recover the Hamel field equations and their compatibility conditions. By combining the discrete Poisson bracket with the symplectic Euler scheme and the implicit midpoint scheme, a Poisson integrator is constructed. Next, using the geometrically exact beam's kinematic model as an example, the reduced Poisson bracket and Hamiltonian equations are derived for both continuous and discrete cases, leading to the Poisson integrator for the geometrically exact beam. Finally, numerical simulations demonstrate that the proposed Poisson integrator preserves energy and momentum while significantly improving computational efficiency compared to the Hamel field integrator.

Key words Hamel's formalism, Poisson integrator, geometric numerical algorithm

引言

几何力学^[1-3]是运用几何观点研究力学系统和场论问题的学科. 其中, Hamel 形式作为 Lagrange 力学的一个版本, 是用位形空间上的活动标架来表

示速度分量. 使用其目的在于充分利用系统的对称性或约束特性来简化动力学方程的表述, 为其动力学与控制问题的求解带来方便. Hamel 形式能够适用于场论问题^[4], 其结合离散 Lagrange 变分力学^[5]得到的 Hamel 场积分子^[6,7]是一种快速、定性

2024-12-02 收到第 1 稿, 2025-01-26 收到修改稿.

^{*} 国家自然科学基金资助项目(12272037, 12232009), National Natural Science Foundation of China (12272037, 12232009).

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准确的数值算法,其已经成功应用于多体动力学与控制研究和实践^[8,9].对于几何精确梁这一实例,已经证明了其拥有保持离散动量的性质^[10].然而,由于Hamel场方程的相容性条件是作为可积性条件单独给出的,用于保证Hamel场方程的解与原Euler-Lagrange方程的解一致,在离散的过程中没有理论上和Hamel场方程统一的离散方式^[11].

将Hamel形式与Hamilton力学相结合,Gao等^[12]在有限维位形流形的情形给出了Hamel方程的Hamilton形式以及对偶Hamel积分分子.场论Hamel形式是基于协变场论的观点结合活动标架方法,然而使用协变Poisson括号得到的协变Hamilton方程组^[13]无法复原Hamel场方程的相容性条件.在非协变观点下,Chadha等^[14-16]给出了考虑翘曲效应的几何精确梁的Poisson括号形式,但未实施对称性约化导致其形式过于复杂.Simo等^[17]给出了三维空间弹性力学系统的Hamilton结构,包括几何精确梁和几何精确壳模型的物质以及对流表示.在离散格式方面,Austin等^[18]证明了对于刚体系统,隐式中点格式二阶精确保持Poisson括号.Demoures等^[19]给出了几何精确梁模型的李群以及李代数积分子,同样未充分利用对称性导致所获算法实现比较复杂.Giusteri等^[20]给出了几何精确梁的有效算法,但未给出相应算法的理论.

本文在Hamilton力学框架下,通过推导Hamel形式下的Poisson括号所得Hamilton方程组能够复原Hamel场方程及其相容性条件,由此通过离散Poisson括号得到Poisson积分子.首先,本文回顾场论Hamel形式,引入对偶标架算子,在此基础上计算得到相应的Poisson括号,由此得到的Hamilton方程组对应Hamel场方程及其相容性条件,进而对两者给出统一的数值格式.接着对于几何精确梁^[21]这一经典算例推导连续方程以及其Poisson积分子,并给出相应的数值结果,说明上述数值格式在不损失保结构特性的同时提高了计算效率.

1 Hamilton系统的Hamel形式

1.1 标架算子^[4]及其对偶

考虑时空可分^[22]的底空间 $B=\mathbb{R}\times S\subset\mathbb{R}^{n+1}$ 上的纤维丛 E .记 $\pi_{BE}:E\rightarrow B$ 为丛投影, $S=\Gamma(\pi_{BE})$

为光滑截面空间,其模型向量空间为 W_Γ ,纤维流形为 F .令 $x^0=t$ 表示时间分量, x^i 表示空间 S 的坐标分量,其中 $i=1,2,\dots,n$.

设 D 为 B 的带边开子流形,取位形空间为其上的光滑局部截面空间 C .给定 $\phi\in C$,标架算子

$$\Psi_\phi:W_\Gamma\rightarrow T_\phi S \quad (1)$$

为光滑依赖于 ϕ 的有界可逆线性算子,其逆 Ψ_ϕ^{-1} 也为光滑有界可逆线性算子.利用对偶关系定义对偶标架算子

$$\Phi_\phi=(\Psi_\phi^*)^{-1}:W_\Gamma^*\rightarrow T_\phi^*S \quad (2)$$

对于任意的 $\xi\in W_\Gamma$, $\Psi_\phi\xi$ 是 ϕ 处的一个切向量,取这些切向量的不交并可得 C 上的向量场

$$\Psi\xi=\bigsqcup_{\phi\in C}\Psi_\phi\xi \quad (3)$$

此处 $\mathfrak{X}(C)$,表示 C 上的向量场空间. $\mathfrak{X}(C)$ 上的Jacobi-Lie括号可以诱导 W_Γ 上的Lie括号 $[\cdot,\cdot]_\phi:W_\Gamma\times W_\Gamma\rightarrow W_\Gamma$ 为

$$[\xi,\eta]_\phi=\Psi_\phi^{-1}[\Psi\xi,\Psi\eta](\phi),\forall\xi,\eta\in W_\Gamma \quad (4)$$

定义对偶括号 $[\cdot,\cdot]_\phi^*:W_\Gamma\times W_\Gamma^*\rightarrow W_\Gamma^*$ 为

$$\langle[\xi,\alpha]_\phi^*,\eta\rangle=\langle\alpha,[\xi,\eta]_\phi\rangle,\forall\alpha\in W_\Gamma^*, \quad (5)$$

此处 $\langle\cdot,\cdot\rangle$ 为 W_Γ^* 与 W_Γ 的配对.记 W_F 为纤维流形 F 的模型向量空间,引入 W_F -值1-形式 ξ 为

$$\xi=\Psi_\phi^{-1}d\phi \quad (6)$$

令 $\Omega^1(D,W_F)$ 为 D 上 W_F -值1-形式空间,此时前述Lie括号可诱导出 $[\cdot,\cdot]_\phi:\Omega^1(D,W_F)\times W_\Gamma\rightarrow\Omega^1(D,W_F)$ 满足

$$[\xi,\eta]_\phi(V)=[\xi(V),\eta]_\phi,\forall\eta\in W_\Gamma,V\in\mathfrak{X}(B) \quad (7)$$

此处 $\mathfrak{X}(B)$ 为 B 上向量场全体.令 ϕ_i 和 $\dot{\phi}_i$ ($i=1,\dots,n$)分别表示 $\phi\in C$ 的时间和空间变化率,则其全微分可以表示为 $d\phi=\phi_i dt+d_0\phi$,其中 $d_0\phi:=\sum_{i=1}^n\phi_i dx^i$ 定义为 ϕ 对于空间变量的全微分.对于 ϕ 的竖直变分 $\delta\phi\in T_\phi C$,其定义了 E 上的竖直向量场,即对于参数曲线 $\phi^\lambda\in C$ 有

$$\delta\phi(x)=\left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0}\phi^\lambda(x)\in V_{\phi(x)}E,\forall x\in D \quad (8)$$

其中 $VE=\ker(T\pi_{BE})$ 为切丛 TE 的竖直子丛.

1.2 Hamel形式下的Poisson括号

在活动标架下,引入前述变量的非物质版本,定义非物质速度 ζ 、非物质应变 γ 以及非物质变分 η 为:

$$\zeta=\Psi_\phi^{-1}\dot{\phi}_i,\gamma=\Psi_\phi^{-1}d_0\phi,\eta=\Psi_\phi^{-1}\delta\phi. \quad (9)$$

由计算易得

$$\delta \boldsymbol{\gamma} = -\boldsymbol{\Psi}_{\phi}^{-1} \mathbf{i}_{\Psi_{\phi} \eta} \mathbf{d} \boldsymbol{\Psi}_{\phi} \boldsymbol{\gamma} + \boldsymbol{\Psi}_{\phi}^{-1} \delta(\phi_i) dx^i, \quad (10)$$

$$\mathbf{d}_0 \eta = -\boldsymbol{\Psi}_{\phi}^{-1} \mathbf{i}_{\Psi_{\phi} \eta} \mathbf{d} \boldsymbol{\Psi}_{\phi} \eta dx^i + \boldsymbol{\Psi}_{\phi}^{-1} \partial_i (\delta \phi) dx^i \quad (11)$$

其中 $\boldsymbol{\gamma}_i = \boldsymbol{\Psi}_{\phi}^{-1} \phi_i$. 由于 $\delta(\phi_i) = (\delta \phi)_i$, 可得变分公式

$$\begin{aligned} \delta \boldsymbol{\gamma} - \mathbf{d}_0 \eta &= -\boldsymbol{\Psi}_{\phi}^{-1} \mathbf{i}_{\Psi_{\phi} \eta} \mathbf{d} \boldsymbol{\Psi}_{\phi} \boldsymbol{\gamma} + \boldsymbol{\Psi}_{\phi}^{-1} \mathbf{i}_{\Psi_{\phi} \eta} \mathbf{d} \boldsymbol{\Psi}_{\phi} \eta dx^i \\ &= \boldsymbol{\Psi}_{\phi}^{-1} [\boldsymbol{\Psi}_{\phi} \boldsymbol{\gamma}, \boldsymbol{\Psi}_{\phi} \eta]_{\phi} = [\boldsymbol{\gamma}, \eta]_{\phi} \end{aligned} \quad (12)$$

这里采用无穷维力学系统的观点, 以下仍用 C 记定义在空间域 $D \cap S$ 上截面全体, ϕ 为其上的运动曲线. 令 $L = L(\phi, \phi_t)$ 表示 Lagrange 函数, 对于弹性力学系统, ϕ 即为形变映射. 由于客观性^[22] 的存在, 可以将势能部分对 ϕ 的依赖局部上转变为对形变梯度 $\mathbf{d}\phi$ 的依赖^[17]. 于是 Lagrange 函数在活动标架下可以表示为 $l = l(\boldsymbol{\gamma}, \zeta)$. 接下来引入动量

$$\mu_{\phi} = \frac{\delta L}{\delta \phi_t} \quad (13)$$

并在对偶标架下, 定义非物质动量

$$\mu = \Phi_{\phi}^{-1} \mu_{\phi} = \Phi_{\phi}^{-1} \frac{\delta L}{\delta \phi_t} \quad (14)$$

于是相应 Hamilton 函数可以表示为 $h(\boldsymbol{\gamma}, \mu) = H(\phi, \mu_{\phi})$. 对于变分公式同理可得

$$\delta \mu = -\Phi_{\phi}^{-1} \mathbf{i}_{\Psi_{\phi} \eta} \mathbf{d} \Phi_{\phi} \mu + \Phi_{\phi}^{-1} \delta \mu_{\phi} \quad (15)$$

在正则坐标下, Marsden 等^[13] 给出了 T^*C 上的 Poisson 括号, 对于任意 $F, G \in \mathfrak{f}(T^*C)$ 有

$$\{F, G\} = \left\langle \frac{\delta F}{\delta \phi}, \frac{\delta G}{\delta \mu_{\phi}} \right\rangle - \left\langle \frac{\delta G}{\delta \phi}, \frac{\delta F}{\delta \mu_{\phi}} \right\rangle \quad (16)$$

其中对偶括号 T_{ϕ}^*C 与 T_{ϕ}^*C 间的配对. 为得到约化后的形式, 对 $f(\boldsymbol{\gamma}, \mu) = F(\phi, \mu_{\phi})$ 边求固定端点变分, 结合分部积分公式以及变分公式(12)和(15)有

$$\begin{aligned} \text{LHS} = \delta f(\boldsymbol{\gamma}, \mu) &= \left\langle \frac{\delta f}{\delta \boldsymbol{\gamma}}, \delta \boldsymbol{\gamma} \right\rangle + \left\langle \delta \mu, \frac{\delta f}{\delta \boldsymbol{\gamma}} \right\rangle \\ &= \left\langle \frac{\delta f}{\delta \boldsymbol{\gamma}}, \mathbf{d}_0 \eta + [\boldsymbol{\gamma}, \eta]_{\phi} \right\rangle + \\ &\quad \left\langle -\Phi_{\phi}^{-1} \mathbf{i}_{\Psi_{\phi} \eta} \mathbf{d} \Phi_{\phi} \mu + \Phi_{\phi}^{-1} \delta \mu_{\phi}, \frac{\delta f}{\delta \boldsymbol{\gamma}} \right\rangle \\ &= \left\langle -\Phi_{\phi} \text{div}_0 \frac{\delta f}{\delta \boldsymbol{\gamma}} + \Phi_{\phi} \left[\boldsymbol{\gamma}, \frac{\delta f}{\delta \boldsymbol{\gamma}} \right]_{\phi}^*, \delta \phi \right\rangle + \\ &\quad \left\langle \delta \mu_{\phi}, \boldsymbol{\Psi}_{\phi} \frac{\delta f}{\delta \mu} \right\rangle + \left\langle \Phi_{\phi} \mu, \mathbf{i}_{\Psi_{\phi} \eta} \mathbf{d} \boldsymbol{\Psi}_{\phi} \left(\frac{\delta f}{\delta \mu} \right) \right\rangle \\ &= \left\langle -\Phi_{\phi} \text{div}_0 \frac{\delta f}{\delta \boldsymbol{\gamma}} + \Phi_{\phi} \left[\boldsymbol{\gamma}, \frac{\delta f}{\delta \boldsymbol{\gamma}} \right]_{\phi}^*, \delta \phi \right\rangle - \end{aligned}$$

$$\begin{aligned} &\left\langle \mathbf{i}_{\Psi_{\phi} \frac{\delta f}{\delta \mu}} \mathbf{d} \Phi_{\phi} \mu + \left[\boldsymbol{\Psi}_{\phi} \frac{\delta f}{\delta \mu}, \Phi_{\phi} \mu \right]_{\phi}^*, \delta \phi \right\rangle + \\ &\left\langle \delta \mu_{\phi}, \boldsymbol{\Psi}_{\phi} \frac{\delta f}{\delta \mu} \right\rangle \end{aligned} \quad (17)$$

$$\text{RHS} = \delta F(\phi, \mu_{\phi}) = \left\langle \frac{\delta F}{\delta \phi}, \delta \phi \right\rangle + \left\langle \delta \mu_{\phi}, \frac{\delta F}{\delta \mu_{\phi}} \right\rangle \quad (18)$$

其中 $\frac{\delta f}{\delta \boldsymbol{\gamma}} \in \Omega^1(D, W_F)^*$, 即 D 上 W_F 值 1-形式空间的对偶空间, 且 $\text{div}_0 \frac{\delta f}{\delta \boldsymbol{\gamma}}$ 由

$$\left\langle \text{div}_0 \frac{\delta f}{\delta \boldsymbol{\gamma}}, \eta \right\rangle = \text{div} \left\langle \frac{\delta f}{\delta \boldsymbol{\gamma}}, \eta \right\rangle - \left\langle \frac{\delta f}{\delta \boldsymbol{\gamma}}, \mathbf{d}_0 \eta \right\rangle \quad (19)$$

给出. 最后由变分基本定理可得变分关系

$$\begin{aligned} \frac{\delta F}{\delta \phi} &= -\Phi_{\phi} \text{div}_0 \frac{\delta f}{\delta \boldsymbol{\gamma}} + \Phi_{\phi} \left[\boldsymbol{\gamma}, \frac{\delta f}{\delta \boldsymbol{\gamma}} \right]_{\phi}^* - \\ &\quad \mathbf{i}_{\Psi_{\phi} \left(\frac{\delta f}{\delta \mu} \right)} \mathbf{d} \Phi_{\phi} \mu - \left[\boldsymbol{\Psi}_{\phi} \frac{\delta f}{\delta \mu}, \Phi_{\phi} \mu \right]_{\phi}^* \end{aligned} \quad (20)$$

$$\frac{\delta F}{\delta \mu_{\phi}} = \boldsymbol{\Psi}_{\phi} \frac{\delta f}{\delta \mu} \quad (21)$$

在活动标架下, 代入变分关系式(20)和(21)可得约化 Poisson 括号

$$\begin{aligned} \{f, g\} &= \left\langle \frac{\delta F}{\delta \phi}, \frac{\delta G}{\delta \mu_{\phi}} \right\rangle - \left\langle \frac{\delta G}{\delta \phi}, \frac{\delta F}{\delta \mu_{\phi}} \right\rangle \\ &= \left\langle \frac{\delta f}{\delta \boldsymbol{\gamma}}, \mathbf{d}_0 \frac{\delta g}{\delta \mu} + \left[\boldsymbol{\gamma}, \frac{\delta g}{\delta \mu} \right]_{\phi} \right\rangle + \\ &\quad \left\langle \left[\frac{\delta g}{\delta \mu}, \mu \right]_{\phi}^* + \text{div}_0 \frac{\delta g}{\delta \boldsymbol{\gamma}} - \left[\boldsymbol{\gamma}, \frac{\delta g}{\delta \boldsymbol{\gamma}} \right]_{\phi}^*, \frac{\delta f}{\delta \mu} \right\rangle \end{aligned} \quad (22)$$

其中 $g(\boldsymbol{\gamma}, \mu) = G(\phi, \mu_{\phi})$. 代入 Hamilton 方程组

$$\dot{f} = \{f, h\} = \left\langle \frac{\delta f}{\delta \boldsymbol{\gamma}}, \dot{\boldsymbol{\gamma}} \right\rangle + \left\langle \dot{\mu}, \frac{\delta f}{\delta \mu} \right\rangle \quad (23)$$

最后比较可得

$$\dot{\boldsymbol{\gamma}} = \mathbf{d}_0 \frac{\delta h}{\delta \mu} + \left[\boldsymbol{\gamma}, \frac{\delta h}{\delta \mu} \right]_{\phi} \quad (24)$$

$$\dot{\mu} = \left[\frac{\delta h}{\delta \mu}, \mu \right]_{\phi}^* + \text{div}_0 \frac{\delta h}{\delta \boldsymbol{\gamma}} - \left[\boldsymbol{\gamma}, \frac{\delta h}{\delta \boldsymbol{\gamma}} \right]_{\phi}^* \quad (25)$$

在经典场论中, 动能一般表示为场的时间导数的二次型, 且能量对动量的导数为场的时间导数. 此时式(25)对应文献[9]中的 Hamel 场方程, 而式(24)对应相容性条件.

2 Poisson 积分子

为简单起见, 考虑空间维数 $n=1$ 的情形. 首先对空间部分进行离散, 设空间网格数为 $J+1$, 步长

为 Δs . 在离散情形中, 记 $\{\phi_i = \phi_j(t)\}_{j=0}^J$ 为 $C^\infty(\mathbb{R}, F)$ 中的序列, 且离散动量和离散应变为

$$\mu_j \in C^\infty(\mathbb{R}, W_F), j=0, \dots, J, \quad (26)$$

$$\gamma_{j+\frac{1}{2}} \in C^\infty(\mathbb{R}, W_F), j=-1, \dots, J \quad (27)$$

于是 t 时刻的状态变量可以表示为

$$z(t) = (\gamma_{-\frac{1}{2}}, \dots, \gamma_{j-\frac{1}{2}}, \mu_j, \gamma_{j+\frac{1}{2}}, \dots, \gamma_{J+\frac{1}{2}}) \quad (28)$$

其中 $\gamma_{-\frac{1}{2}}$ 和 $\gamma_{J+\frac{1}{2}}$ 由边界条件给出, 这里不妨设为 0. 由离散变分公式^[9]

$$\delta\gamma_{j+\frac{1}{2}} = \frac{1}{\Delta s}(\eta_{j+1} - \eta_j) + \left[\gamma_{j+\frac{1}{2}}, \frac{1}{2}(\eta_{j+1} + \eta_j) \right] \quad (29)$$

$$\delta\mu_j = -\Phi_{\phi_j}^{-1} \mathbf{i}_{\Psi_{\phi_j}} \eta_j d\Phi_{\phi_j} \mu_j + \Phi_{\phi_j}^{-1} \delta(\mu_{\phi_j})_j \quad (30)$$

结合离散变分原理可得离散变分关系

$$\begin{aligned} \Psi_{\phi_j}^* \frac{\delta F}{\delta \phi_j} = & -\frac{1}{\Delta s} \left(\frac{\delta f}{\delta \gamma_{j+\frac{1}{2}}} - \frac{\delta f}{\delta \gamma_{j-\frac{1}{2}}} \right) + \\ & \frac{1}{2} \left(\left[\gamma_{j+\frac{1}{2}}, \frac{\delta f}{\delta \gamma_{j+\frac{1}{2}}} \right]^* + \left[\gamma_{j-\frac{1}{2}}, \frac{\delta f}{\delta \gamma_{j-\frac{1}{2}}} \right]^* \right) - \\ & \Phi_{\phi_j}^{-1} \mathbf{i}_{\Psi_{\phi_j}} \left(\frac{\delta f}{\delta \mu_j} \right) d\Phi_{\phi_j} \mu_j - \Phi_{\phi_j}^{-1} \left[\Psi_{\phi_j} \frac{\delta f}{\delta \mu_j}, \Phi_{\phi_j} \mu_j \right]^* \end{aligned} \quad (31)$$

$$\frac{\delta F}{\delta(\mu_{\phi_j})_j} = \Psi_{\phi_j} \left(\frac{\delta f}{\delta \mu_j} \right) \quad (32)$$

故离散约化 Poisson 括号为

$$\begin{aligned} \{f, h\}_d = & \sum_{j=0}^{J-1} \left\langle \frac{\delta f}{\delta \gamma_{j+\frac{1}{2}}}, \frac{1}{\Delta s} \left(\frac{\delta h}{\delta \mu_{j+1}} - \frac{\delta h}{\delta \mu_j} \right) \right\rangle + \\ & \sum_{j=0}^{J-1} \left\langle \frac{\delta f}{\delta \gamma_{j+\frac{1}{2}}}, \left[\gamma_{j+\frac{1}{2}}, \frac{1}{2} \left(\frac{\delta h}{\delta \mu_{j+1}} + \frac{\delta h}{\delta \mu_j} \right) \right] \right\rangle + \\ & \sum_{j=1}^{J-1} \left\langle \frac{1}{\Delta s} \left(\frac{\delta h}{\delta \gamma_{j+\frac{1}{2}}} - \frac{\delta h}{\delta \gamma_{j-\frac{1}{2}}} \right), \frac{\delta f}{\delta \mu_j} \right\rangle - \\ & \sum_{j=1}^{J-1} \left\langle \frac{1}{2} \left[\gamma_{j+\frac{1}{2}}, \frac{\delta h}{\delta \gamma_{j+\frac{1}{2}}} \right]^*, \frac{\delta f}{\delta \mu_j} \right\rangle - \\ & \sum_{j=1}^{J-1} \left\langle \frac{1}{2} \left[\gamma_{j-\frac{1}{2}}, \frac{\delta h}{\delta \gamma_{j-\frac{1}{2}}} \right]^*, \frac{\delta f}{\delta \mu_j} \right\rangle - \\ & \left\langle \frac{1}{\Delta s} \frac{\delta h}{\delta \gamma_{J-\frac{1}{2}}} + \frac{1}{2} \left[\gamma_{J-\frac{1}{2}}, \frac{\delta h}{\delta \gamma_{J-\frac{1}{2}}} \right]^*, \frac{\delta f}{\delta \mu_J} \right\rangle - \\ & \left\langle \frac{1}{\Delta s} \frac{\delta h}{\delta \gamma_{\frac{1}{2}}} + \frac{1}{2} \left[\gamma_{\frac{1}{2}}, \frac{\delta h}{\delta \gamma_{\frac{1}{2}}} \right]^*, \frac{\delta f}{\delta \mu_0} \right\rangle + \\ & \sum_{j=0}^J \left\langle \left[\frac{\delta h}{\delta \mu_j}, \mu_j \right]^*, \frac{\delta f}{\delta \mu_j} \right\rangle \end{aligned} \quad (33)$$

同理得空间离散的 Hamilton 方程组

$$\begin{aligned} \dot{\mu}_j = & \frac{1}{\Delta s} \left(\frac{\delta h}{\delta \gamma_{j+\frac{1}{2}}} - \frac{\delta h}{\delta \gamma_{j-\frac{1}{2}}} \right) - \frac{1}{2} \left(\left[\gamma_{j+\frac{1}{2}}, \frac{\delta h}{\delta \gamma_{j+\frac{1}{2}}} \right]^* + \right. \\ & \left. \left[\gamma_{j-\frac{1}{2}}, \frac{\delta h}{\delta \gamma_{j-\frac{1}{2}}} \right]^* \right) + \left[\frac{\delta h}{\delta \mu_j}, \mu_j \right]^* \end{aligned} \quad (34)$$

$$\begin{aligned} \dot{\gamma}_{j+\frac{1}{2}} = & \frac{1}{\Delta s} \left(\frac{\delta h}{\delta \mu_{j+1}} - \frac{\delta h}{\delta \mu_j} \right) + \\ & \left[\gamma_{j+\frac{1}{2}}, \frac{1}{2} \left(\frac{\delta h}{\delta \mu_{j+1}} + \frac{\delta h}{\delta \mu_j} \right) \right] \end{aligned} \quad (35)$$

设时间网格数为 $N+1$, 步长为 Δt . 记 $\{\phi_j^n = \phi_j(t_n)\}_{j=0, \dots, J}^{n=0, \dots, N}$, 对于 μ_j^n 、 γ_{j+1}^n 等同理. 对于刚体系统, Austin 等^[18]证明了隐式中点格式二阶精确保持 Poisson 括号. 在此基础上, 为避免像李群变分积分分子^[19,23]或 Giusteri 等^[20]一样求解大型线性方程组, 结合辛 Euler 格式^[24]可得如下数值格式

$$\begin{aligned} \frac{1}{\Delta t}(\mu_j^{n+1} - \mu_j^n) = & \frac{1}{\Delta s} \left(\frac{\delta h}{\delta \gamma_{j+\frac{1}{2}}^n} - \frac{\delta h}{\delta \gamma_{j-\frac{1}{2}}^n} \right) - \\ & \frac{1}{2} \left(\left[\gamma_{j+\frac{1}{2}}^n, \frac{\delta h}{\delta \gamma_{j+\frac{1}{2}}^n} \right]^* + \left[\gamma_{j-\frac{1}{2}}^n, \frac{\delta h}{\delta \gamma_{j-\frac{1}{2}}^n} \right]^* \right) + \\ & \left[\frac{1}{2} \left(\frac{\delta h}{\delta \mu_j^{n+1}} + \frac{\delta h}{\delta \mu_j^n} \right), \frac{1}{2}(\mu_j^{n+1} + \mu_j^n) \right]^*, \\ \frac{1}{\Delta t}(\gamma_{j+\frac{1}{2}}^{n+1} - \gamma_{j+\frac{1}{2}}^n) = & \frac{1}{\Delta s} \left(\frac{\delta h}{\delta \mu_{j+1}^{n+1}} - \frac{\delta h}{\delta \mu_j^{n+1}} \right) + \\ & \left[\frac{1}{2}(\gamma_{j+\frac{1}{2}}^{n+1} + \gamma_{j+\frac{1}{2}}^n), \frac{1}{2} \left(\frac{\delta h}{\delta \mu_{j+1}^{n+1}} + \frac{\delta h}{\delta \mu_j^{n+1}} \right) \right] \end{aligned} \quad (36)$$

3 几何精确梁的动力学模型及其 Poisson 积分

在几何精确梁的动力学模型^[21]中, 首先引入右手正交标架 $\{O, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, 称为参考标架. 在参考标架下, 初始时梁的中轴线位于 E_2 轴上. 假设梁的截面做刚体运动.

考虑底空间 $B = \mathbb{R} \times [0, 1]$ 上的主丛 $E = B \times G$, 丛投影为 $\pi_{BE}: E \rightarrow B$, 纤维流形等于结构群 $G = SE(3)$, 其中 l 为梁长. 于是梁的位形空间为丛的光滑截面空间 $C = C^\infty(B, E)$. 此外由于李群 $SE(3)$ 的李代数 $\mathfrak{se}(3)$ 同构于 $\mathbb{R}^6$, 后文中将默认使用这一关系.

对于几何精确梁模型, 通过标架算子 Ψ 的选取可实现 Lie-Poisson 约化^[25]. 在活动标架下定义非物质速度以及非物质应变为

$$\zeta = \Psi_g^{-1} g_t = g^{-1} g_t \quad (37)$$

$$\gamma = \Psi_g^{-1} g_s = g^{-1} g_s - e_2 \quad (38)$$

其中 $\mathbf{e}_2 = (0, \mathbf{E}_2)$ 为 $\mathbb{R}^6$ 中的单位向量. 几何精确梁的 Lagrange 函数具有 SE(3) 不变性, 可以表示为

$$l(\boldsymbol{\zeta}, \boldsymbol{\gamma}) = \frac{1}{2} \langle \mathbf{D}_1 \boldsymbol{\zeta}, \boldsymbol{\zeta} \rangle - \frac{1}{2} \langle \mathbf{D}_2 \boldsymbol{\gamma}, \boldsymbol{\gamma} \rangle \quad (39)$$

其中 $\langle \cdot, \cdot \rangle$ 为 L^2 配对, $\mathbf{D}_1$ 和 $\mathbf{D}_2$ 为与梁的材料有关的常正定矩阵^[9]. 由 Legendre 变换 $\boldsymbol{\mu} = (\delta l) / (\delta \boldsymbol{\zeta})$, Hamilton 函数可以表示为

$$h(\boldsymbol{\gamma}, \boldsymbol{\mu}) = \frac{1}{2} \langle \boldsymbol{\mu}, \mathbf{D}_1^{-1} \boldsymbol{\mu} \rangle + \frac{1}{2} \langle \mathbf{D}_2 \boldsymbol{\gamma}, \boldsymbol{\gamma} \rangle \quad (40)$$

由计算易得约化 Poisson 括号为

$$\begin{aligned} \{f, h\} = & \left\langle \frac{\delta f}{\delta \boldsymbol{\gamma}}, \partial_s \frac{\delta h}{\delta \boldsymbol{\mu}} + \left[\boldsymbol{\gamma} + \mathbf{e}_2, \frac{\delta h}{\delta \boldsymbol{\mu}} \right]_g \right\rangle + \\ & \left\langle \frac{\delta f}{\delta \boldsymbol{\mu}}, \partial_s \frac{\delta h}{\delta \boldsymbol{\gamma}} - \left[\boldsymbol{\gamma} + \mathbf{e}_2, \frac{\delta h}{\delta \boldsymbol{\gamma}} \right]_g + \left[\frac{\delta h}{\delta \boldsymbol{\mu}}, \boldsymbol{\mu} \right]_g^* \right\rangle \end{aligned} \quad (41)$$

最后代入约化 Hamilton 函数可得

$$\frac{\delta h}{\delta \boldsymbol{\mu}} = \mathbf{D}_1^{-1} \boldsymbol{\mu} = \boldsymbol{\zeta}, \quad \frac{\delta h}{\delta \boldsymbol{\gamma}} = \mathbf{D}_2 \boldsymbol{\gamma} \quad (42)$$

于是对应的 Hamilton 方程组为

$$\dot{\boldsymbol{\gamma}} = \partial_s \boldsymbol{\zeta} + [\boldsymbol{\gamma} + \mathbf{e}_2, \boldsymbol{\zeta}]_g \quad (43)$$

$$\dot{\boldsymbol{\mu}} = [\boldsymbol{\zeta}, \boldsymbol{\mu}]_g^* + \partial_s (\mathbf{D}_2 \boldsymbol{\gamma}) - [\boldsymbol{\gamma} + \mathbf{e}_2, \mathbf{D}_2 \boldsymbol{\gamma}]_g^* \quad (44)$$

与文[9]中结果一致. 给定边界条件

$$\mathbf{D}_2 \boldsymbol{\gamma}(0) = \mathbf{D}_2 \boldsymbol{\gamma}(l) = 0 \quad (45)$$

设时间和空间步长分别为 Δt 和 Δs , 时间和空间网格数分别为 $N+1$ 和 $J+1$. 令

$$\mathbf{z}_n = (\boldsymbol{\gamma}_{-\frac{1}{2}}^n, \dots, \boldsymbol{\gamma}_{j-\frac{1}{2}}^n, \boldsymbol{\mu}_j^n, \boldsymbol{\gamma}_{j+\frac{1}{2}}^n, \dots, \boldsymbol{\gamma}_{J+\frac{1}{2}}^n) \quad (46)$$

表示 $t = t_n$ 时刻的状态变量, 其中离散非物质动量和离散非物质应变为

$$\boldsymbol{\mu}_j^n \in \mathbb{R}^6, j = 0, \dots, J \quad (47)$$

$$\boldsymbol{\gamma}_{j+\frac{1}{2}}^n \in \mathbb{R}^6, j = -1, \dots, J \quad (48)$$

离散约化 Hamilton 函数可以表示为

$$\begin{aligned} h^n = h(\boldsymbol{\gamma}^n) = & \sum_{j=0}^J \frac{1}{2} \langle \boldsymbol{\mu}_j^n, \mathbf{D}_1^{-1} \boldsymbol{\mu}_j^n \rangle + \\ & \sum_{j=-1}^J \frac{1}{2} \langle \mathbf{D}_2 \boldsymbol{\gamma}_{j+\frac{1}{2}}^n, \boldsymbol{\gamma}_{j+\frac{1}{2}}^n \rangle \end{aligned} \quad (49)$$

由计算可得离散约化 Poisson 括号为

$$\begin{aligned} \{f, h\}_d = & \sum_{j=0}^{J-1} \left\langle \frac{\delta f}{\delta \boldsymbol{\gamma}_{j+\frac{1}{2}}}, \frac{1}{\Delta s} \left(\frac{\delta h}{\delta \boldsymbol{\mu}_{j+1}} - \frac{\delta h}{\delta \boldsymbol{\mu}_j} \right) + \right. \\ & \left. \left[\boldsymbol{\gamma}_{j+\frac{1}{2}} + \mathbf{e}_2, \frac{1}{2} \left(\frac{\delta h}{\delta \boldsymbol{\mu}_{j+1}} + \frac{\delta h}{\delta \boldsymbol{\mu}_j} \right) \right] \right\rangle - \\ & \sum_{j=0}^J \left\langle -\frac{1}{\Delta s} \left(\frac{\delta h}{\delta \boldsymbol{\gamma}_{j+\frac{1}{2}}} - \frac{\delta h}{\delta \boldsymbol{\gamma}_{j-\frac{1}{2}}} \right), \frac{\delta f}{\delta \boldsymbol{\mu}_j} \right\rangle - \end{aligned}$$

$$\begin{aligned} & \sum_{j=0}^J \left\langle \frac{1}{2} \left(\left[\boldsymbol{\gamma}_{j+\frac{1}{2}} + \mathbf{e}_2, \frac{\delta h}{\delta \boldsymbol{\gamma}_{j+\frac{1}{2}}} \right]^* + \right. \right. \\ & \left. \left. \left[\boldsymbol{\gamma}_{j-\frac{1}{2}} + \mathbf{e}_2, \frac{\delta h}{\delta \boldsymbol{\gamma}_{j-\frac{1}{2}}} \right]^* \right), \frac{\delta f}{\delta \boldsymbol{\mu}_j} \right\rangle + \\ & \left\langle \frac{1}{\Delta s} \frac{\delta h}{\delta \boldsymbol{\mu}_0} - \frac{1}{2} \left[\boldsymbol{\gamma}_{-\frac{1}{2}} + \mathbf{e}_2, \frac{\delta h}{\delta \boldsymbol{\mu}_0} \right]^*, \frac{\delta f}{\delta \boldsymbol{\gamma}_{-\frac{1}{2}}} \right\rangle - \\ & \left\langle \frac{1}{\Delta s} \frac{\delta h}{\delta \boldsymbol{\mu}_J} + \frac{1}{2} \left[\boldsymbol{\gamma}_{J+\frac{1}{2}} + \mathbf{e}_2, \frac{\delta h}{\delta \boldsymbol{\mu}_J} \right]^*, \frac{\delta f}{\delta \boldsymbol{\gamma}_{J+\frac{1}{2}}} \right\rangle + \\ & \sum_{j=0}^J \left\langle \left[\frac{\delta h}{\delta \boldsymbol{\mu}_j}, \boldsymbol{\mu}_j \right]^*, \frac{\delta f}{\delta \boldsymbol{\mu}_j} \right\rangle \end{aligned} \quad (50)$$

故空间离散的 Hamilton 方程组(34)与(35)表示为

$$\begin{aligned} \dot{\boldsymbol{\mu}}_j = & \frac{1}{\Delta s} \left(\frac{\delta h}{\delta \boldsymbol{\gamma}_{j+\frac{1}{2}}} - \frac{\delta h}{\delta \boldsymbol{\gamma}_{j-\frac{1}{2}}} \right) + \left[\frac{\delta h}{\delta \boldsymbol{\mu}_j}, \boldsymbol{\mu}_j \right]^* - \\ & \frac{1}{2} \left(\left[\boldsymbol{\gamma}_{j+\frac{1}{2}} + \mathbf{e}_2, \frac{\delta h}{\delta \boldsymbol{\gamma}_{j+\frac{1}{2}}} \right]^* + \left[\boldsymbol{\gamma}_{j-\frac{1}{2}} + \mathbf{e}_2, \frac{\delta h}{\delta \boldsymbol{\gamma}_{j-\frac{1}{2}}} \right]^* \right) \end{aligned} \quad (51)$$

$$\begin{aligned} \dot{\boldsymbol{\gamma}}_{j+\frac{1}{2}} = & \frac{1}{\Delta s} \left(\frac{\delta h}{\delta \boldsymbol{\mu}_{j+1}} - \frac{\delta h}{\delta \boldsymbol{\mu}_j} \right) + \\ & \left[\boldsymbol{\gamma}_{j+\frac{1}{2}} + \mathbf{e}_2, \frac{1}{2} \left(\frac{\delta h}{\delta \boldsymbol{\mu}_{j+1}} + \frac{\delta h}{\delta \boldsymbol{\mu}_j} \right) \right] \end{aligned} \quad (52)$$

与式(36)对应的数值格式如下

$$\begin{aligned} \frac{1}{\Delta t} (\boldsymbol{\mu}_{j+1}^{n+1} - \boldsymbol{\mu}_j^n) = & \frac{1}{\Delta s} (\mathbf{D}_2 \boldsymbol{\gamma}_{j+\frac{1}{2}}^n - \mathbf{D}_2 \boldsymbol{\gamma}_{j-\frac{1}{2}}^n) + \\ & \left[\frac{1}{2} (\boldsymbol{\zeta}_{j+1}^{n+1} + \boldsymbol{\zeta}_j^n), \frac{1}{2} (\boldsymbol{\mu}_{j+1}^{n+1} + \boldsymbol{\mu}_j^n) \right]^* - \\ & \frac{1}{2} ([\boldsymbol{\gamma}_{j+\frac{1}{2}}^n + \mathbf{e}_2, \mathbf{D}_2 \boldsymbol{\gamma}_{j+\frac{1}{2}}^n]^* + \\ & [\boldsymbol{\gamma}_{j-\frac{1}{2}}^n + \mathbf{e}_2, \mathbf{D}_2 \boldsymbol{\gamma}_{j-\frac{1}{2}}^n]^*) \end{aligned} \quad (53)$$

$$\begin{aligned} \frac{1}{\Delta t} (\boldsymbol{\gamma}_{j+\frac{1}{2}}^{n+1} - \boldsymbol{\gamma}_{j+\frac{1}{2}}^n) = & \frac{1}{\Delta s} (\boldsymbol{\zeta}_{j+1}^{n+1} - \boldsymbol{\zeta}_j^{n+1}) + \\ & \left[\frac{1}{2} (\boldsymbol{\gamma}_{j+\frac{1}{2}}^{n+1} + \boldsymbol{\gamma}_{j+\frac{1}{2}}^n + 2\mathbf{e}_2), \frac{1}{2} (\boldsymbol{\zeta}_{j+1}^{n+1} + \boldsymbol{\zeta}_j^{n+1}) \right] \end{aligned} \quad (54)$$

在得到某一时刻的离散非物质动量后, 通过 Cayley 变换^[26]重构梁的位形, 即

$$\begin{aligned} \mathbf{g}_j^{n+1} \approx & \mathbf{g}_j^n \text{cay}(\Delta t \mathbf{D}_1^{-1} \boldsymbol{\mu}_j^n) \\ = & \mathbf{g}_j^n \left(\mathbf{I}_4 - \frac{\Delta t}{2} \mathbf{D}_1^{-1} \boldsymbol{\mu}_j^n \right)^{-1} \left(\mathbf{I}_4 + \frac{\Delta t}{2} \mathbf{D}_1^{-1} \boldsymbol{\mu}_j^n \right) \end{aligned} \quad (55)$$

其中 $\mathbf{I}_4$ 为四阶单位矩阵.

综上结合边界条件(45)可以得到数值算法如下:

Algorithm 1 几何精确梁的 Poisson 积分分子

Input:

迭代次数 T ;

初始离散非物质动量集合 $\{\boldsymbol{\mu}_j^0\}_{j=0}^J$;

初始离散非物质应变集合 $\{\boldsymbol{\gamma}_{j+1/2}^0\}_{j=0}^J$;

Output:

梁的位形 $\{g_j^n\}_{n=1,\cdots,T,j=0,\cdots,J}$

```
1: for each  $n \in [1, T]$  do
2:   for each  $j \in [0, J]$  do
3:     使用 Newton 法求解非线性方程组 (53)
4:   end for
5:   for each  $j \in [0, J]$  do
6:     求解线性方程组 (54)
7:   end for
8:   使用 Cayley 变换恢复该时刻节点处梁的位形
    $\{g_j^n\}_{j=0}^J$ 
9: end for
```

4 数值仿真

考虑不受外力作用的几何精确梁,即使用自由边界条件

$$\boldsymbol{\gamma}_{-\frac{1}{2}}^n = \boldsymbol{\gamma}_{J+\frac{1}{2}}^n = 0, \forall n = 0, \cdots, N \tag{56}$$

取时间步长为 0.000 1 s,空间网格数为 101.

初始离散非物质动量与应变为

$$\left\{ \boldsymbol{\mu}_j^0 = \left(0, 0, 0, a^2 \rho \sin\left(\frac{j\pi}{100}\right), a^2 \rho, 0 \right) \right\}_{j=0}^{100},$$

$$\left\{ \boldsymbol{\gamma}_{j+\frac{1}{2}}^0 = (-1, 0, 0, 0, 0, 0) \right\}_{j=0}^{99}$$

其中梁的参数^[9]如表 1 所示.

表 1 梁的参数
Table 1 Beam Parameters

名称	符号	值
梁长	l	$2\pi/3$
横截面边长	a	0.1
密度	ρ	100 0
杨氏模量	E	10 000 000
泊松比	ν	0.35

数值结果如图 1~3 所示.

从图 1 可以看出,使用上述数值格式能够将梁的总能量保持在振动幅度为 1 的范围内.图 2 和图 3 则说明以上数值格式具有在 0.000 1 精度下保持线动量和角动量的性质.这两条性质与几何精确梁的 Hamel 场变分积分子^[9]差异不大.经过格式对比发现,从半离散格式的角度来看,以上数值格式相当于将 Hamel 场变分积分子中关于时间的梯形

格式变成了中点格式.这样做的好处是能够减少数值格式中括号项的格式从而减少计算量.

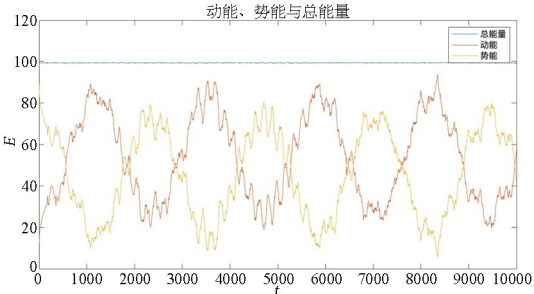


图 1 动能、势能和总能量
Fig. 1 Kinetic, Potential and Total Energy

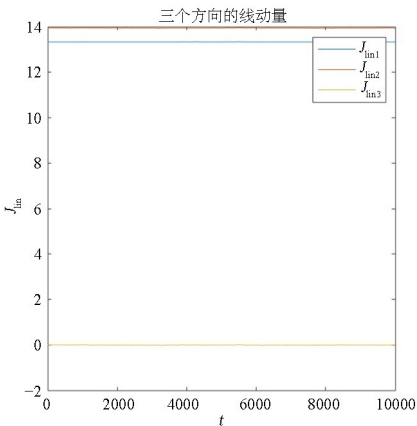


图 2 线动量
Fig. 2 Linear Momentum

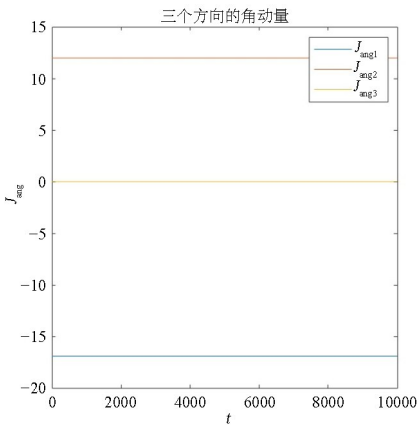


图 3 角动量
Fig. 3 Angular Momentum

在比较以上两种数值格式的计算效率时,使用 MATLAB 和 Python 这两种语言编程.由表 2,在相同计算资源条件下比较两种数值格式迭代 1 000 个时间步长所用计算时间,发现以上数值格式在计算效率上分别提升了 3.22%(MATLAB)和 14.49%(Python).

综上所述,以上数值格式在保持保结构特性的同时一定程度上提升了计算效率.

表 2 1 000 次迭代所用计算时间
Table 2 Computation time for 1 000 iterations

方法	Poisson 积分分子	Hamel 积分分子
MATLAB	13.554 3 s	14.005 4 s
Python	15.947 0 s	18.628 5 s

5 结论

本文对具对称性的无穷维 Hamilton 系统使用活动标架实现 Lie-Poisson 约化,对于几何精确梁这一实例得到了能够复原 Hamel 场方程及其相容性条件的 Hamlton 方程组.由此提出几何精确梁的 Poisson 积分分子,并且通过数值仿真说明该算法具有保持能量、动量的特点.相较于传统的 Hamel 场积分分子^[9],这里采用无穷维动力系统观点,同时离散 Hamel 方程及其相容性条件,得到一致的离散格式.

通过对几何精确梁的数值仿真,说明该算法在保持能量以及动量的同时提高了计算效率.下一步将把此算法推广到空间高维场论以及多物理场的情形.

参考文献

[1] ABRAHAM R, MARSDEN J. Foundations of mechanics, second edition [M]. Redding, MA, USA: Addison-Wesley, 1978.

[2] ARNOLD V I. Mathematical methods of classical mechanics [M]. New York: Springer New York, 1989.

[3] SOURIAU J M. Structure des systèmes dynamiques: maîtrises de mathématiques [M]. Cham, Switzerland:Springer, 1970.

[4] SHI D H, ZENKOV D V, BLOCH A M. Hamel's formalism for classical field theories [J]. Journal of Nonlinear Science, 2020, 30(4): 1307—1353.

[5] MARSDEN J E, WEST M. Discrete mechanics and variational integrators [J]. Acta Numerica, 2001, 10: 357—514.

[6] BALL K R. Structure preserving integrators and Hamel's equations [D]. Raleigh, NC, USA: North Carolina State University, 2013.

[7] BALL K R, ZENKOV D V. Hamel's formalism and variational integrators [M]//CHANG D D, HOLM D D, Patrick G, et al. Geometry, Mecha-

tics, and Dynamics. New York: Springer New York, 2015:477—506, 2015.

[8] 沈伟清, 王本亮, 史东华. 开链机械臂的反馈 Hamel 变分积分分子[J]. 动力学与控制学报, 2022, 20(3): 8—14.

SHEN W Q, WANG B L, SHI D H. Feedback Hamel's variational integrators of open-chain manipulators [J]. Journal of Dynamics and Control, 2022, 20(3): 8—14. (in Chinese)

[9] 王亮, 安志朋, 史东华. 几何精确梁的 Hamel 场变分积分分子[J]. 北京大学学报(自然科学版), 2016, 52(4): 692—698.

WANG L, AN Z P, SHI D H. Hamel's field variational integrator of geometrically exact beam [J]. Acta Scientiarum Naturalium Universitatis Pekinensis, 2016, 52(4): 692—698. (in Chinese)

[10] 高山, 史东华, 郭永新. Hamel 框架下几何精确梁的离散动量守恒律[J]. 力学学报, 2021, 53(6): 1712—1719.

GAO S, SHI D H, GUO Y X. Discrete momentum conservation law of geometrically exact beam in hamel's framework [J]. Chinese Journal of Theoretical and Applied Mechanics, 2021, 53(6): 1712—1719. (in Chinese)

[11] AN Z P, GAO S, SHI D H, et al. A variational integrator for the chaplygin-Timoshenko sleigh [J]. Journal of Nonlinear Science, 2020, 30(4): 1381—1419.

[12] GAO S, SHI D H, ZENKOV D V. Discrete Hamiltonian variational mechanics and Hamel's integrators [J]. Journal of Nonlinear Science, 33(2):26, 2023.

[13] MARSDEN J E, MONTGOMERY R, MORRISON P J, et al. Covariant Poisson brackets for classical fields [J]. Annals of Physics, 1986, 169(1):29—47.

[14] CHADHA M, TODD M D. Poisson bracket formulation of a higher-order, geometrically-exact beam [J]. Applied Mathematics Letters, 2021, 113: 106842.

[15] CHADHA M, TODD M D. A comprehensive kinematic model of single-manifold Cosserat beam structures with application to a finite strain measurement model for strain gauges [J]. International Journal of Solids and Structures, 2019, 159: 58—76.

[16] CHADHA M, TODD M D. The mathematical theory of a higher-order geometrically-exact beam with a deforming cross-section [J]. International Journal

- of Solids and Structures, 2020, 202: 854–880.
- [17] SIMO J C, MARSDEN J E, KRISHNAPRASAD P S. The Hamiltonian structure of nonlinear elasticity: the material and convective representations of solids, rods, and plates [J]. Archive for Rational Mechanics and Analysis, 1988, 104(2): 125–183.
- [18] AUSTIN M A, KRISHNAPRASAD P S, WANG L S. Almost Poisson integration of rigid body systems [J]. Journal of Computational Physics, 1993, 107(1): 105–117.
- [19] DEMOURES F, GAY-BALMAZ F, LEYEN-DECKER S, et al. Discrete variational Lie group formulation of geometrically exact beam dynamics [J]. Numerische Mathematik, 2015, 130(1): 73–123.
- [20] GIUSTERI G G, MIGLIO E, PAROLINI N, et al. Simulation of viscoelastic Cosserat rods based on the geometrically exact dynamics of special Euclidean strands [J]. International Journal for Numerical Methods in Engineering, 2022, 123(2): 396–410.
- [21] AURICCHIOA F, CAROTENUTO P, REALI A. On the geometrically exact beam model: A consistent, effective and simple derivation from three-dimensional finite-elasticity [J]. International Journal of Solids and Structures, 2008, 45(17): 4766–4781.
- [22] Marsden J E, Hughes Thomas J R. Mathematical foundations of elasticity [M]. Englewood Cliffs, NJ, USA: Prentice-Hall, 1983.
- [23] 黄子恒, 陈菊, 张志娟, 等. 多刚体动力学仿真的李群变分积分算法[J]. 动力学与控制学报, 2022, 20(1): 8–17.
- HUANG Z H, CHEN J, ZHANG Z J, et al. Lie group variational integration for multi-rigid body system dynamic simulation [J]. Journal of Dynamics and Control, 2022, 20(1): 8–17. (in Chinese)
- [24] HAIRER E, HOCHBRUCK M, ISELES A, et al. Geometric numerical integration [J]. Oberwolfach Reports, 2006, 3(1): 805–882.
- [25] MARSDEN J, RATIU T. Introduction to mechanics and symmetry [M]. New York: Springer New York, 1999.
- [26] LEWIS D, SIMO J C. Conserving algorithms for the dynamics of Hamiltonian systems on lie groups [J]. Journal of Nonlinear Science, 1994, 4(1): 253–299.

附录

A: 离散 Poisson 括号计算

由离散变分公式(29)和(30),有

$$\begin{aligned}
 & \sum_{j=0}^J \left\langle \frac{\delta F}{\delta \phi_j}, \delta \phi_j \right\rangle + \sum_{j=0}^J \left\langle \delta(\mu_\phi)_j, \left(\frac{\delta F}{\delta \mu_\phi} \right)_j \right\rangle = \sum_{j=0}^{J-1} \left\langle \frac{\delta f}{\delta \gamma_{j+\frac{1}{2}}}, \delta \gamma_{j+\frac{1}{2}} \right\rangle + \sum_{j=0}^J \left\langle \delta \mu_j, \frac{\delta f}{\delta \mu_j} \right\rangle \\
 & = \sum_{j=0}^{J-1} \left\langle \frac{\delta f}{\delta \gamma_{j+\frac{1}{2}}}, \frac{1}{\Delta s} (\eta_{j+1} - \eta_j) + \left[\gamma_{j+\frac{1}{2}}, \frac{1}{2} (\eta_{j+1} + \eta_j) \right] \right\rangle + \sum_{j=0}^J \left\langle -\Phi_{\phi_j}^{-1} \mathbf{i}_{\Psi_{\phi_j} \eta_j} \mathbf{d}\Phi_{\phi_j} \mu_j + \Phi_{\phi_j}^{-1} \delta(\mu_\phi)_j, \frac{\delta f}{\delta \mu_j} \right\rangle \\
 & = \sum_{j=1}^{J-1} \left\langle -\frac{1}{\Delta s} \left(\frac{\delta f}{\delta \gamma_{j+\frac{1}{2}}} - \frac{\delta f}{\delta \gamma_{j-\frac{1}{2}}} \right) + \frac{1}{2} \left(\left[\gamma_{j-\frac{1}{2}}, \frac{\delta f}{\delta \gamma_{j-\frac{1}{2}}} \right]^* + \left[\gamma_{j+\frac{1}{2}}, \frac{\delta f}{\delta \gamma_{j+\frac{1}{2}}} \right]^* \right), \eta_j \right\rangle + \left\langle \frac{1}{\Delta s} \frac{\delta f}{\delta \gamma_{J-\frac{1}{2}}} + \right. \\
 & \quad \left. \frac{1}{2} \left[\gamma_{J-\frac{1}{2}}, \frac{\delta f}{\delta \gamma_{J-\frac{1}{2}}} \right]^*, \eta_J \right\rangle + \left\langle -\frac{1}{\Delta s} \left(\frac{\delta f}{\delta \gamma} \right)_{\frac{1}{2}} + \frac{1}{2} \left[\gamma_{\frac{1}{2}}, \left(\frac{\delta f}{\delta \gamma} \right)_{\frac{1}{2}} \right]^*, \eta_0 \right\rangle + \sum_{j=0}^J \left\langle -\mathbf{i}_{\Psi_{\phi_j} \eta_j} \mathbf{d}\Phi_{\phi_j} \mu_j, \Psi_{\phi_j} \frac{\delta f}{\delta \mu_j} \right\rangle + \\
 & \quad \sum_{j=0}^J \left\langle \delta(\mu_\phi)_j, \Psi_{\phi_j} \frac{\delta f}{\delta \mu_j} \right\rangle = \sum_{j=1}^{J-1} \left\langle -\frac{1}{\Delta s} \left(\frac{\delta f}{\delta \gamma_{j+\frac{1}{2}}} - \frac{\delta f}{\delta \gamma_{j-\frac{1}{2}}} \right) + \frac{1}{2} \left(\left[\gamma_{j-\frac{1}{2}}, \frac{\delta f}{\delta \gamma_{j-\frac{1}{2}}} \right]^* + \left[\gamma_{j+\frac{1}{2}}, \frac{\delta f}{\delta \gamma_{j+\frac{1}{2}}} \right]^* \right), \eta_j \right\rangle + \\
 & \quad \left\langle \frac{1}{\Delta s} \frac{\delta f}{\delta \gamma_{J-\frac{1}{2}}} + \frac{1}{2} \left[\gamma_{J-\frac{1}{2}}, \frac{\delta f}{\delta \gamma_{J-\frac{1}{2}}} \right]^*, \eta_J \right\rangle + \left\langle -\frac{1}{\Delta s} \left(\frac{\delta f}{\delta \gamma} \right)_{\frac{1}{2}} + \frac{1}{2} \left[\gamma_{\frac{1}{2}}, \left(\frac{\delta f}{\delta \gamma} \right)_{\frac{1}{2}} \right]^*, \eta_0 \right\rangle + \\
 & \quad \sum_{j=0}^J \left\langle -\Phi_{\phi_j}^{-1} \mathbf{i}_{\Psi_{\phi_j} \eta_j} \mathbf{d}\Phi_{\phi_j} \mu_j - \Phi_{\phi_j}^{-1} \left[\Psi_{\phi_j} \frac{\delta f}{\delta \mu_j}, \Phi_{\phi_j} \mu_j \right]^*, \eta_j \right\rangle + \sum_{j=0}^J \left\langle \delta(\mu_\phi)_j, \Psi_{\phi_j} \frac{\delta f}{\delta \mu_j} \right\rangle
 \end{aligned}$$

其中最后一个等号由分部积分公式以及式(12),边界积分项为零,即

由变分基本定理,有

代入 Poisson 括号可得

$$\begin{aligned}
\{f, h\} &= \sum_{j=0}^J \left\langle \frac{\delta F}{\delta \phi_j}, \frac{\delta H}{\delta (\mu_\phi)_j} \right\rangle - \left\langle \frac{\delta H}{\delta \phi_j}, \frac{\delta F}{\delta (\mu_\phi)_j} \right\rangle = \sum_{j=1}^{J-1} \left\langle -\frac{1}{\Delta s} \left(\frac{\delta f}{\delta \gamma_{j+\frac{1}{2}}} - \frac{\delta f}{\delta \gamma_{j-\frac{1}{2}}} \right) + \right. \\
&\quad \left. \frac{1}{2} \left(\left[\gamma_{j+\frac{1}{2}}^n, \frac{\delta f}{\delta \gamma_{j+\frac{1}{2}}} \right]^* + \left[\gamma_{j-\frac{1}{2}}, \frac{\delta f}{\delta \gamma_{j-\frac{1}{2}}} \right]^* \right), \frac{\delta h}{\delta \mu_j} \right\rangle + \left\langle \frac{1}{\Delta s} \frac{\delta f}{\delta \gamma_{J-\frac{1}{2}}} + \frac{1}{2} \left[\gamma_{J-\frac{1}{2}}, \frac{\delta f}{\delta \gamma_{J-\frac{1}{2}}} \right]^*, \frac{\delta h}{\delta \mu_J} \right\rangle + \\
&\quad \left\langle -\frac{1}{\Delta s} \frac{\delta f}{\delta \gamma_{\frac{1}{2}}} + \frac{1}{2} \left[\gamma_{\frac{1}{2}}, \frac{\delta f}{\delta \gamma_{\frac{1}{2}}} \right]^*, \frac{\delta h}{\delta \mu_0} \right\rangle + \sum_{j=0}^J \left\langle -\Phi_{\phi_j}^{-1} \mathbf{i}_{\Psi_{\phi_j} \left(\frac{\delta f}{\delta \mu_j} \right)} \mathbf{d} \Phi_{\phi_j} \mu_j - \Phi_{\phi_j}^{-1} \left[\Psi_{\phi_j} \frac{\delta f}{\delta \mu_j}, \Phi_{\phi_j} \mu_j \right]^*, \frac{\delta h}{\delta \mu_j} \right\rangle - \\
&\quad \sum_{j=1}^{J-1} \left\langle -\frac{1}{\Delta s} \left(\frac{\delta h}{\delta \gamma_{j+\frac{1}{2}}} - \frac{\delta h}{\delta \gamma_{j-\frac{1}{2}}} \right) + \frac{1}{2} \left(\left[\gamma_{j+\frac{1}{2}}^n, \frac{\delta h}{\delta \gamma_{j+\frac{1}{2}}} \right]^* + \left[\gamma_{j-\frac{1}{2}}, \frac{\delta h}{\delta \gamma_{j-\frac{1}{2}}} \right]^* \right), \frac{\delta f}{\delta \mu_j} \right\rangle - \left\langle \frac{1}{\Delta s} \frac{\delta h}{\delta \gamma_{J-\frac{1}{2}}} + \right. \\
&\quad \left. \frac{1}{2} \left[\gamma_{J-\frac{1}{2}}, \frac{\delta h}{\delta \gamma_{J-\frac{1}{2}}} \right]^*, \frac{\delta f}{\delta \mu_J} \right\rangle - \left\langle -\frac{1}{\Delta s} \frac{\delta h}{\delta \gamma_{\frac{1}{2}}} + \frac{1}{2} \left[\gamma_{\frac{1}{2}}, \frac{\delta h}{\delta \gamma_{\frac{1}{2}}} \right]^*, \frac{\delta f}{\delta \mu_0} \right\rangle - \sum_{j=0}^J \left\langle -\Phi_{\phi_j}^{-1} \mathbf{i}_{\Psi_{\phi_j} \left(\frac{\delta h}{\delta \mu_j} \right)} \mathbf{d} \Phi_{\phi_j} \mu_j - \right. \\
&\quad \left. \Phi_{\phi_j}^{-1} \left[\Psi_{\phi_j} \frac{\delta h}{\delta \mu_j}, \Phi_{\phi_j} \mu_j \right]^*, \frac{\delta f}{\delta \mu_j} \right\rangle \\
&= \sum_{j=0}^{J-1} \left\langle \frac{\delta f}{\delta \gamma_{j+\frac{1}{2}}}, \frac{1}{\Delta s} \left(\frac{\delta h}{\delta \mu_{j+1}} - \frac{\delta h}{\delta \mu_j} \right) + \left[\gamma_{j+\frac{1}{2}}, \frac{1}{2} \left(\frac{\delta h}{\delta \mu_{j+1}} + \frac{\delta h}{\delta \mu_j} \right) \right] \right\rangle + \sum_{j=0}^J \left\langle \Phi_{\phi_j} \mu_j, \mathbf{i}_{\Psi_{\phi_j} \left(\frac{\delta f}{\delta \mu_j} \right)} \mathbf{d} \Psi_{\phi_j} \frac{\delta h}{\delta \mu_j} - \right. \\
&\quad \left[\Psi_{\phi_j} \frac{\delta f}{\delta \mu_j}, \Psi_{\phi_j} \frac{\delta h}{\delta \mu_j} \right] \rangle - \sum_{j=1}^{J-1} \left\langle -\frac{1}{\Delta s} \left(\frac{\delta h}{\delta \gamma_{j+\frac{1}{2}}} - \frac{\delta h}{\delta \gamma_{j-\frac{1}{2}}} \right) + \frac{1}{2} \left(\left[\gamma_{j+\frac{1}{2}}^n, \frac{\delta h}{\delta \gamma_{j+\frac{1}{2}}} \right]^* + \left[\gamma_{j-\frac{1}{2}}, \frac{\delta h}{\delta \gamma_{j-\frac{1}{2}}} \right]^* \right), \frac{\delta f}{\delta \mu_j} \right\rangle - \\
&\quad \left\langle \frac{1}{\Delta s} \frac{\delta h}{\delta \gamma_{J-\frac{1}{2}}} + \frac{1}{2} \left[\gamma_{J-\frac{1}{2}}, \frac{\delta h}{\delta \gamma_{J-\frac{1}{2}}} \right]^*, \frac{\delta f}{\delta \mu_J} \right\rangle - \left\langle -\frac{1}{\Delta s} \frac{\delta h}{\delta \gamma_{\frac{1}{2}}} + \frac{1}{2} \left[\gamma_{\frac{1}{2}}, \frac{\delta h}{\delta \gamma_{\frac{1}{2}}} \right]^*, \frac{\delta f}{\delta \mu_0} \right\rangle - \\
&\quad \sum_{j=0}^J \left\langle \Phi_{\phi_j} \mu_j, \mathbf{i}_{\Psi_{\phi_j} \left(\frac{\delta h}{\delta \mu_j} \right)} \mathbf{d} \Psi_{\phi_j} \frac{\delta f}{\delta \mu_j} - \left[\Psi_{\phi_j} \frac{\delta h}{\delta \mu_j}, \Psi_{\phi_j} \frac{\delta f}{\delta \mu_j} \right] \right\rangle \\
&= \sum_{j=0}^{J-1} \left\langle \frac{\delta f}{\delta \gamma_{j+\frac{1}{2}}}, \frac{1}{\Delta s} \left(\frac{\delta h}{\delta \mu_{j+1}} - \frac{\delta h}{\delta \mu_j} \right) + \left[\gamma_{j+\frac{1}{2}}, \frac{1}{2} \left(\frac{\delta h}{\delta \mu_{j+1}} + \frac{\delta h}{\delta \mu_j} \right) \right] \right\rangle - \sum_{j=1}^{J-1} \left\langle -\frac{1}{\Delta s} \left(\frac{\delta h}{\delta \gamma_{j+\frac{1}{2}}} - \frac{\delta h}{\delta \gamma_{j-\frac{1}{2}}} \right) + \right. \\
&\quad \left. \frac{1}{2} \left(\left[\gamma_{j+\frac{1}{2}}^n, \frac{\delta h}{\delta \gamma_{j+\frac{1}{2}}} \right]^* + \left[\gamma_{j-\frac{1}{2}}, \frac{\delta h}{\delta \gamma_{j-\frac{1}{2}}} \right]^* \right), \frac{\delta f}{\delta \mu_j} \right\rangle - \left\langle \frac{1}{\Delta s} \frac{\delta h}{\delta \gamma_{J-\frac{1}{2}}} + \frac{1}{2} \left[\gamma_{J-\frac{1}{2}}, \frac{\delta h}{\delta \gamma_{J-\frac{1}{2}}} \right]^*, \frac{\delta f}{\delta \mu_J} \right\rangle -
\end{aligned}$$

$$\left\langle -\frac{1}{\Delta s} \frac{\delta h}{\delta \gamma_{\frac{1}{2}}} + \frac{1}{2} \left[\gamma_{\frac{1}{2}}, \frac{\delta h}{\delta \gamma_{\frac{1}{2}}} \right]^* \cdot \frac{\delta f}{\delta \mu_0} \right\rangle + \sum_{j=0}^J \left\langle \left[\frac{\delta h}{\delta \mu_j}, \mu_j \right]^* \cdot \frac{\delta f}{\delta \mu_j} \right\rangle$$