

二阶非完整系统 Vacco 动力学的 Herglotz 型 Noether 定理^{*}

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摘要 研究二阶非完整系统 Vacco 动力学的 Herglotz 型 Noether 定理. 首先, 基于 Herglotz 广义变分原理, 建立了二阶非完整系统 Vacco 动力学的 Herglotz 型运动微分方程. 其次, 根据 Hamilton-Herglotz 作用量的非等时变分公式, 给出了二阶非完整系统 Vacco 动力学的 Herglotz 型 Noether 对称性和准对称性的概念及其判据方程, 并推导了系统的 Herglotz 型 Noether 定理及其逆定理. 最后, 举例说明了该结果的应用.

关键词 二阶非完整系统, Vacco 动力学, Herglotz 变分原理, Noether 定理

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Herglotz-Type Noether Theorem for Vacco Dynamics of Second-Order Nonholonomic Systems^{*}

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Abstract The Herglotz-type Noether theorems for the Vacco dynamics of second-order nonholonomic systems are studied. Firstly, based on the Herglotz generalized variational principle, the Herglotz-type differential equations of motion for Vacco dynamics of second-order nonholonomic systems are established. Secondly, according to the non-isochronous variation formulas of Hamilton-Herglotz action, the concepts of Herglotz-type Noether symmetry and quasi-symmetry and their criterion equations for Vacco dynamics of second-order nonholonomic systems are given, and the Herglotz-type Noether theorems and their inverse theorems are further derived. Finally, an example is given to illustrate the application of the results.

Key words second-order nonholonomic systems, Vacco dynamics, Herglotz variational principle, Noether's theorem

引言

非完整力学系统作为分析力学的重要分支, 其理论广泛应用于轮式系统、电机系统等领域^[1]. 梅凤翔先生曾指出: 非完整系统动力学允许有多种模型, 约束的物理实现不是唯一的^[2]. 非完整力学模

型主要有 Chetaev 型和 Vacco 型, 分别以 d'Alembert-Lagrange 原理和嵌入约束的 Hamilton 原理为基础, 前者是力学的, 后者是数学的^[3]. Vacco 动力学是由 Kozlov^[4] 提出的研究不可积分约束系统动力学的一种新的数学模型, 主要用于研究控制问题. 动力学系统对称性与守恒量的研究在现代数

理科学中占有重要地位^[5]. 近几十年来,关于 Vacco 动力学及其对称性的研究已经取得了较多成果,如 Vacco 动力学的几何理论^[6,7]、Noether 对称性^[8,9]、Mei 对称性^[10]、Lie 对称性^[11]和高阶 Vacco 动力学^[12-14]等. 陈立群教授^[12]在研究高阶非完整系统 Vacco 动力学时,还证明了微分算子 d 和变分算子 δ 顺序可交换. 然而,迄今关于 Vacco 动力学的研究很少涉及受高阶非完整约束的非保守系统. Herglotz^[15]在研究接触变换及其与 Poisson 括号和 Hamilton 系统的关系时曾提出一类新的变分原理,该原理作为经典变分原理的推广可用于解决非保守问题. 2002 年,Georgieva 等^[16]利用 Lagrange 函数在无穷小变换下的不变性建立了 Herglotz 型 Noether 定理. 此后,许多学者对 Herglotz 变分原理及其 Noether 定理展开了深入研究^[17-28]. 2015 年,Santos 等从最优控制的角度研究了 Herglotz 型高阶变分问题^[29]及其 Noether 定理^[30]. 迄今为止,关于非保守系统的 Herglotz 型 Noether 守恒量的研究已经取得了重要进展,但研究仍局限于完整系统或一阶非完整系统层面. 然而,由于机器人系统的动力学控制和自动控制技术发展的需要,二阶或更高阶非完整约束系统动力学得到人们的关注^[31-35]. 基于此,本文将从 Herglotz 变分原理出发,在文献[13]的基础上,研究二阶非完整非保守系统 Vacco 动力学的 Herglotz 型运动微分方程及其 Noether 理论.

为方便计算,文中采用爱因斯坦求和约定.

1 二阶非完整 Vacco 系统动力学的 Herglotz 型运动微分方程

研究 N 个质点构成的力学系统,其位形由 n 个广义坐标 $q_s (s=1,2,\dots,n)$ 确定,假设系统的运动受到 g 个独立的二阶非完整约束

$$f_\beta(t, q_s, \dot{q}_s, \ddot{q}_s) = 0, (\beta=1,2,\dots,g) \quad (1)$$

下面研究二阶非完整系统 Vacco 动力学的 Herglotz 变分问题.

确定正轨 $q_s(t) \in C^2([t_0, t_1], \mathbb{R})$, 由微分方程

$$\dot{z}(t) = L[t, q_s(t), \dot{q}_s(t), z(t)] + \lambda_\beta(t) f_\beta[t, q_s(t), \dot{q}_s(t), \ddot{q}_s(t)] \quad (2)$$

定义的作用量 $z(t)$ 在给定的边界条件

$$q_s(t) \big|_{t=t_0} = q_{s0}, q_s(t) \big|_{t=t_1} = q_{s1} \quad (3)$$

和初始条件

$$z(t) \big|_{t=t_0} = z_{t_0} \quad (4)$$

$z(t_1)$ 取得极值. 其中, $\lambda_\beta = \lambda_\beta(t)$ 为引入的 Lagrange 乘子, $L(t, q_s, \dot{q}_s, z)$ 是 Herglotz 型 Lagrange 函数. q_{s0}, q_{s1}, z_{t_0} 均为常数.

方程(2)两端同时进行等时变分运算,可得

$$\delta \dot{z} = \frac{\partial L}{\partial q_s} \delta q_s + \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s + \frac{\partial L}{\partial z} \delta z + \lambda_\beta \left(\frac{\partial f_\beta}{\partial q_s} \delta q_s + \frac{\partial f_\beta}{\partial \dot{q}_s} \delta \dot{q}_s + \frac{\partial f_\beta}{\partial \ddot{q}_s} \delta \ddot{q}_s \right) \quad (5)$$

根据文献[12],对于二阶非线性非完整系统,满足微分运算 d 和变分运算 δ 的顺序可交换. 因此,我们解上述方程,可得

$$\begin{aligned} & \delta z(t) \exp\left(-\int_{t_0}^t \frac{\partial L}{\partial z} d\theta\right) - \delta z(t_0) \\ &= \int_{t_0}^t \left[\left(\frac{\partial L}{\partial q_s} + \lambda_\beta \frac{\partial f_\beta}{\partial q_s} \right) \delta q_s + \left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) \delta \dot{q}_s + \lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} \delta \ddot{q}_s \right] \exp\left(-\int_{t_0}^\tau \frac{\partial L}{\partial z} d\theta\right) d\tau \end{aligned} \quad (6)$$

因为 $z(t_1)$ 有极值,而在 $t=t_0$ 时满足条件(4),故有

$$\delta z(t) \big|_{t=t_0} = \delta z(t) \big|_{t=t_1} = 0 \quad (7)$$

方程(6)两端同时取 $t=t_1$, 并利用式(7)可得

$$\begin{aligned} & \int_{t_0}^{t_1} \left[\left(\frac{\partial L}{\partial q_s} + \lambda_\beta \frac{\partial f_\beta}{\partial q_s} \right) \delta q_s + \left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) \delta \dot{q}_s + \lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} \delta \ddot{q}_s \right] \exp\left(-\int_{t_0}^\tau \frac{\partial L}{\partial z} d\theta\right) d\tau = 0 \end{aligned} \quad (8)$$

由分部积分法可知

$$\begin{aligned} & \int_{t_0}^{t_1} \delta \dot{q}_s \left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) \exp\left(-\int_{t_0}^\tau \frac{\partial L}{\partial z} d\theta\right) d\tau \\ &= \left[\delta q_s \left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) \exp\left(-\int_{t_0}^\tau \frac{\partial L}{\partial z} d\theta\right) \right] \bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1} \delta q_s \frac{d}{d\tau} \left[\left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) \exp\left(-\int_{t_0}^\tau \frac{\partial L}{\partial z} d\theta\right) \right] d\tau \end{aligned} \quad (9)$$

以及

$$\begin{aligned} & \int_{t_0}^{t_1} \lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} \delta \ddot{q}_s \exp\left(-\int_{t_0}^\tau \frac{\partial L}{\partial z} d\theta\right) d\tau \\ &= \left[\lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} \delta \dot{q}_s \exp\left(-\int_{t_0}^\tau \frac{\partial L}{\partial z} d\theta\right) \right] \bigg|_{t_0}^{t_1} - \left\{ \delta q_s \frac{d}{d\tau} \left[\lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} \exp\left(-\int_{t_0}^\tau \frac{\partial L}{\partial z} d\theta\right) \right] \right\} \bigg|_{t_0}^{t_1} + \int_{t_0}^{t_1} \delta q_s \frac{d^2}{d\tau^2} \left[\lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} \exp\left(-\int_{t_0}^\tau \frac{\partial L}{\partial z} d\theta\right) \right] d\tau \end{aligned} \quad (10)$$

将式(9)、式(10)代入式(8),并利用式(3),我们有

$$\int_{t_0}^{t_1} \delta q_s \left\{ \left(\frac{\partial L}{\partial q_s} + \lambda_\beta \frac{\partial f_\beta}{\partial q_s} \right) \cdot \exp\left(-\int_{t_0}^\tau \frac{\partial L}{\partial z} d\theta\right) - \right.$$

$$\frac{d}{dt} \left[\left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] + \frac{d^2}{dt^2} \left[\lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] dt = 0 \quad (11)$$

根据 Lagrange 乘子法^[36], 且由于式(11)对任意积分区间 $[t_0, t_1]$ 都满足, 故有

$$\exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \left(\frac{\partial L}{\partial q_s} + \lambda_\beta \frac{\partial f_\beta}{\partial q_s} \right) - \frac{d}{dt} \left[\left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] + \frac{d^2}{dt^2} \left[\lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] = 0 \quad (12)$$

我们称式(12)为二阶非完整系统 Vacco 动力学的 Herglotz 型运动微分方程。

2 二阶非完整 Vacco 动力学的 Herglotz 型 Noether 对称性

引入无穷小变换为

$$t^* = t + \Delta t, q_s^*(t^*) = q_s(t) + \Delta q_s(t) \quad (13)$$

或其展开式

$$t^* = t + \epsilon_\sigma \xi_0^\sigma(t, q_k, \dot{q}_k, z), \\ q_s^*(t^*) = q_s(t) + \epsilon_\sigma \xi_s^\sigma(t, q_k, \dot{q}_k, z) \quad (14)$$

其中 $\xi_0^\sigma, \xi_s^\sigma (s=1, 2, \dots, n)$ 为无穷小生成元, 而 $\epsilon_\sigma (\sigma=1, 2, \dots, r)$ 无穷小参数。

由于泛函 $z(t)$ 经过变换式(13)成为 $z^*(t^*)$, 它们之间的关系如下

$$z^*(t^*) = z(t) + \Delta z(t) \quad (15)$$

且对任意一个可微函数 $F(t)$, 有^[37]

$$\Delta \dot{F} = \frac{d}{dt} \Delta F - \dot{F} \frac{d}{dt} \Delta t \quad (16)$$

从而有

$$\frac{d}{dt} \Delta z = \left(\frac{\partial L}{\partial t} + \lambda_\beta \frac{\partial f_\beta}{\partial t} \right) \Delta t + \left(\frac{\partial L}{\partial q_s} + \lambda_\beta \frac{\partial f_\beta}{\partial q_s} \right) \Delta q_s + \frac{\partial L}{\partial z} \Delta z + \left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) \Delta \dot{q}_s + \lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} \Delta \ddot{q}_s + \dot{\lambda}_\beta f_\beta \Delta t + (L + \lambda_\beta f_\beta) \frac{d}{dt} \Delta t \quad (17)$$

积分之, 可得

$$\Delta z(t) \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) - \Delta z(t_0) = \int_{t_0}^t \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \left[\left(\frac{\partial L}{\partial t} + \lambda_\beta \frac{\partial f_\beta}{\partial t} \right) \Delta t + \left(\frac{\partial L}{\partial q_s} + \right. \right.$$

$$\left. \lambda_\beta \frac{\partial f_\beta}{\partial q_s} \right) \Delta q_s + \left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) \Delta \dot{q}_s + \lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} \Delta \ddot{q}_s + \dot{\lambda}_\beta f_\beta \Delta t + (L + \lambda_\beta f_\beta) \frac{d}{dt} \Delta t \right] dt \quad (18)$$

由于 $\Delta z(t_0) = 0$, 则式(18)还可以写成

$$\Delta z(t) \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) = \int_{t_0}^t \frac{d}{dt} \left\{ \left[\left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) (\Delta q_s - \dot{q}_s \Delta t) + (L + \lambda_\beta f_\beta) \cdot \Delta t + \lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} (\Delta \dot{q}_s - \ddot{q}_s \Delta t) \right] \cdot \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) - \frac{d}{dt} \left[\lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] (\Delta q_s - \dot{q}_s \Delta t) \right\} + \left\{ \left(\frac{\partial L}{\partial q_s} + \lambda_\beta \frac{\partial f_\beta}{\partial q_s} \right) \cdot \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) - \frac{d}{dt} \left[\left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] + \frac{d^2}{dt^2} \left[\lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] \right\} (\Delta q_s - \dot{q}_s \Delta t) dt \quad (19)$$

将生成元 ξ_0^σ 和 ξ_s^σ 代入式(18)和式(19), 可得

$$\Delta z(t) \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) = \int_{t_0}^t \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \cdot \left[\left(\frac{\partial L}{\partial t} + \lambda_\beta \frac{\partial f_\beta}{\partial t} \right) \xi_0^\sigma + \left(\frac{\partial L}{\partial q_s} + \lambda_\beta \frac{\partial f_\beta}{\partial q_s} \right) \xi_s^\sigma + (L + \lambda_\beta f_\beta) \dot{\xi}_0^\sigma + \left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) (\dot{\xi}_s^\sigma - \dot{q}_s \xi_0^\sigma) + \dot{\lambda}_\beta f_\beta \xi_0^\sigma + \lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} (\ddot{\xi}_s^\sigma - 2\ddot{q}_s \dot{\xi}_0^\sigma - \dot{q}_s \ddot{\xi}_0^\sigma) \right] \epsilon_\sigma dt \quad (20)$$

以及

$$\Delta z(t) \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) = \int_{t_0}^t \epsilon_\sigma \frac{d}{dt} \left\{ \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \cdot \left[\left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) \bar{\xi}_s^\sigma + (L + \lambda_\beta f_\beta) \xi_0^\sigma + \lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} (\dot{\xi}_s^\sigma - \dot{q}_s \dot{\xi}_0^\sigma - \ddot{q}_s \xi_0^\sigma) \right] - \frac{d}{dt} \left[\lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] \bar{\xi}_s^\sigma \right\} + \epsilon_\sigma \left\{ \left(\frac{\partial L}{\partial q_s} + \lambda_\beta \frac{\partial f_\beta}{\partial q_s} \right) \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) - \frac{d}{dt} \left[\left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] + \right.$$

$$\frac{d^2}{dt^2} \left[\lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] \bar{\xi}_s^\sigma dt \quad (21)$$

其中 $\bar{\xi}_s^\sigma = \xi_s^\sigma - \dot{q}_s \xi_0^\sigma$. 方程(20)和(21)是二阶非完整系统 Vacco 动力学的 Hamilton-Herglotz 作用量变分的两个基本公式.

如果对于系统的 Hamilton-Herglotz 作用量泛函 z , 有

$$\Delta z(t_1) = 0 \quad (22)$$

成立, 则称变换式(14)为 Herglotz 型 Noether 意义下的对称变换.

此时根据式(20)可得

$$\begin{aligned} & \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \cdot \left[\left(\frac{\partial L}{\partial t} + \lambda_\beta \frac{\partial f_\beta}{\partial t} \right) \xi_0^\sigma + \right. \\ & \left(\frac{\partial L}{\partial q_s} + \lambda_\beta \frac{\partial f_\beta}{\partial q_s} \right) \xi_s^\sigma + (L + \lambda_\beta f_\beta) \dot{\xi}_0^\sigma + \\ & \left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) (\dot{\xi}_s^\sigma - \dot{q}_s \dot{\xi}_0^\sigma) + \dot{\lambda}_\beta f_\beta \xi_0^\sigma + \\ & \left. \lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} (\ddot{\xi}_s^\sigma - 2\ddot{q}_s \dot{\xi}_0^\sigma - \dot{q}_s \ddot{\xi}_0^\sigma) \right] = 0 \end{aligned} \quad (23)$$

由二阶非完整约束方程(1)和 Vacco 动力学的 Herglotz 型方程(12)确定的系统

$$\begin{aligned} & \Delta z(t_1) \exp \left(- \int_{t_0}^{t_1} \frac{\partial L}{\partial z} d\theta \right) \\ & = \int_{t_0}^{t_1} \frac{d}{dt} \left[\epsilon_s G^\sigma \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] dt \end{aligned} \quad (24)$$

其中 $G^\sigma = G^\sigma(t, q_s, \dot{q}_s, z)$ 为规范函数, 则称变换式(14)为 Herglotz 型 Noether 意义下的准对称变换.

此时由方程(20)得到

$$\begin{aligned} & \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \left[\left(\frac{\partial L}{\partial t} + \lambda_\beta \frac{\partial f_\beta}{\partial t} \right) \xi_0^\sigma + \dot{\lambda}_\beta f_\beta \xi_0^\sigma + \right. \\ & \left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) (\dot{\xi}_s^\sigma - \dot{q}_s \dot{\xi}_0^\sigma) + (L + \lambda_\beta f_\beta) \dot{\xi}_0^\sigma + \\ & \left. \lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} (\ddot{\xi}_s^\sigma - 2\ddot{q}_s \dot{\xi}_0^\sigma - \dot{q}_s \ddot{\xi}_0^\sigma) + \left(\frac{\partial L}{\partial q_s} + \right. \right. \\ & \left. \left. \lambda_\beta \frac{\partial f_\beta}{\partial q_s} \right) \xi_s^\sigma \right] = \frac{d}{dt} \left[G^\sigma \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] \end{aligned} \quad (25)$$

式(23)和式(25)称为二阶非完整系统 Vacco 动力学的 Herglotz 型 Noether 判据方程.

3 二阶非完整 Vacco 动力学的 Herglotz 型 Noether 定理

在对称变换下, 由式(21)和二阶非完整 Vacco 动力学的 Herglotz 型方程(12)可得

$$\begin{aligned} & \frac{d}{dt} \left\{ \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \cdot \left[\left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) \bar{\xi}_s^\sigma + \right. \right. \\ & (L + \lambda_\beta f_\beta) \xi_0^\sigma + \lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} (\dot{\xi}_s^\sigma - \dot{q}_s \dot{\xi}_0^\sigma - \ddot{q}_s \xi_0^\sigma) \left. \right] - \\ & \left. \frac{d}{dt} \left[\lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] \bar{\xi}_s^\sigma \right\} = 0 \end{aligned} \quad (26)$$

对上式进行积分可得 Herglotz 型 Noether 守恒量

$$\begin{aligned} & \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \cdot \left[\left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) \bar{\xi}_s^\sigma + \right. \\ & (L + \lambda_\beta f_\beta) \xi_0^\sigma + \lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} (\dot{\xi}_s^\sigma - \dot{q}_s \dot{\xi}_0^\sigma - \ddot{q}_s \xi_0^\sigma) \left. \right] - \\ & \frac{d}{dt} \left[\lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] \bar{\xi}_s^\sigma = \text{const.} \end{aligned} \quad (27)$$

在准对称变换下, 还可得 Herglotz 型 Noether 守恒量

$$\begin{aligned} & \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \cdot \left[\left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \right) \bar{\xi}_s^\sigma - G^\sigma + \right. \\ & (L + \lambda_\beta f_\beta) \xi_0^\sigma + \lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} (\dot{\xi}_s^\sigma - \dot{q}_s \dot{\xi}_0^\sigma - \ddot{q}_s \xi_0^\sigma) \left. \right] - \\ & \frac{d}{dt} \left[\lambda_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] \bar{\xi}_s^\sigma = \text{const.} \end{aligned} \quad (28)$$

定理 1 如果对于由二阶非完整约束方程(1)和 Vacco 动力学的 Herglotz 型方程(12)确定的系统, 生成元函数 $\xi_0^\sigma, \xi_s^\sigma$ 和规范函数 G^σ 满足 Noether 判据方程(25), 则此系统存在 Herglotz 型 Noether 守恒量式(28).

特别地, 当 $G^\sigma \equiv 0$ 时, 上述定理成为:

定理 2 若对于由二阶非完整约束方程(1)和 Vacco 动力学的 Herglotz 型方程确定的系统, 生成元函数 $\xi_0^\sigma, \xi_s^\sigma$ 满足 Noether 判据方程(23), 则此系统存在 Herglotz 型 Noether 守恒量式(27).

定理 1 和定理 2 为二阶非完整系统 Vacco 动力学的 Herglotz 型 Noether 定理. 若 $L(t, q_s, \dot{q}_s, z)$ 不再依赖于 $z(t)$, 即满足 $\partial L(t, q_s, \dot{q}_s, z)/\partial z = 0$, 则退化为经典 Lagrange 函数 $L(t, q_s, \dot{q}_s)$, 此时二阶非完整系统 Vacco 动力学的 Herglotz 型方程(12)将退化为^[12,13]:

$$\begin{aligned} & \frac{\partial L}{\partial q_s} + \lambda_\beta \frac{\partial f_\beta}{\partial q_s} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} - \lambda_\beta \frac{d}{dt} \frac{\partial f_\beta}{\partial \dot{q}_s} - \dot{\lambda}_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} + \\ & \ddot{\lambda}_\beta \frac{\partial f_\beta}{\partial \ddot{q}_s} + 2\dot{\lambda}_\beta \frac{d}{dt} \left(\frac{\partial f_\beta}{\partial \ddot{q}_s} \right) + \lambda_\beta \frac{d^2}{dt^2} \left(\frac{\partial f_\beta}{\partial \ddot{q}_s} \right) = 0 \end{aligned} \quad (29)$$

特别地, 若约束是一阶非完整的, 则定理 2 退化为二阶非完整系统 Vacco 动力学的 Herglotz 型

Noether 定理;若 Herglotz 型 Lagrange 函数同时也不含作用量 z , 那么定理 2 将退化为经典的 Vacco 动力学系统的 Noether 定理^[8], 即为如下推论.

推论 1 如果对于一阶非完整 Vacco 动力学系统, 变换式(14)是 Noether 准对称变换, 则该系统存在 r 个线性独立的第一积分, 形如

$$(L + \lambda_{\beta} f_{\beta}) \bar{\xi}_0^{\sigma} + \left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_{\beta} \frac{\partial f_{\beta}}{\partial \dot{q}_s} \right) \bar{\xi}_s^{\sigma} - G^{\sigma} = \text{const.}, (\sigma = 1, 2, \dots, r) \quad (30)$$

反过来, 已知二阶非完整 Vacco 动力学系统的一个第一积分为

$$I^{\sigma}(t, q_i, \dot{q}_i, z) = \text{const.}, (\sigma = 1, 2, \dots, r) \quad (31)$$

对式(31)两端关于 t 求导, 并与式(12)两端同时乘以 $\bar{\xi}_s^{\sigma}$ 后相加, 可得

$$\begin{aligned} \frac{dI^{\sigma}}{dt} + \left\{ \left(\frac{\partial L}{\partial q_s} + \lambda_{\beta} \frac{\partial f_{\beta}}{\partial q_s} \right) \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) - \right. \\ \left. \frac{d}{dt} \left[\left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_{\beta} \frac{\partial f_{\beta}}{\partial \dot{q}_s} \right) \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] + \right. \\ \left. \frac{d^2}{dt^2} \left[\lambda_{\beta} \frac{\partial f_{\beta}}{\partial \ddot{q}_s} \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] \right\} \bar{\xi}_s^{\sigma} = 0 \quad (32) \end{aligned}$$

令 \ddot{q}_s 项的系数为 0, 即

$$\begin{aligned} \frac{\partial I^{\sigma}}{\partial \dot{q}_i} - \left(\frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_i} + \lambda_{\beta} \frac{\partial^2 f_{\beta}}{\partial \dot{q}_s \partial \dot{q}_i} + 2\lambda_{\beta} \frac{\partial L}{\partial z} \frac{\partial^2 f_{\beta}}{\partial \ddot{q}_s \partial \dot{q}_i} + \right. \\ \left. \lambda_{\beta} \frac{\partial f_{\beta}}{\partial \ddot{q}_s} \frac{\partial^2 L}{\partial z \partial \dot{q}_i} - \lambda_{\beta} \frac{\partial^3 f_{\beta}}{\partial \ddot{q}_s \partial \dot{q}_i \partial t} - \dot{\lambda}_{\beta} \frac{\partial^3 f_{\beta}}{\partial \ddot{q}_s \partial t \partial \dot{q}_i} - \right. \\ \left. \lambda_{\beta} \frac{\partial^2 f_{\beta}}{\partial \ddot{q}_s \partial q_i} - 2\dot{\lambda}_{\beta} \frac{\partial^2 f_{\beta}}{\partial \ddot{q}_s \partial \dot{q}_i} \right) \exp \left(- \int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \bar{\xi}_s^{\sigma} = 0 \quad (33) \end{aligned}$$

从上述方程反解出 $\bar{\xi}_s^{\sigma}$ 可得

$$\bar{\xi}_s^{\sigma} = \tilde{b}_{si} \cdot \exp \left(\int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \cdot \frac{\partial I^{\sigma}}{\partial \dot{q}_i} \quad (34)$$

其中

$$\begin{aligned} b_{si} = \frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_i} + \lambda_{\beta} \frac{\partial^2 f_{\beta}}{\partial \dot{q}_s \partial \dot{q}_i} + 2\lambda_{\beta} \frac{\partial L}{\partial z} \cdot \frac{\partial^2 f_{\beta}}{\partial \ddot{q}_s \partial \dot{q}_i} + \\ \lambda_{\beta} \frac{\partial f_{\beta}}{\partial \ddot{q}_s} \cdot \frac{\partial^2 L}{\partial z \partial \dot{q}_i} - 2\dot{\lambda}_{\beta} \frac{\partial^2 f_{\beta}}{\partial \ddot{q}_s \partial \dot{q}_i} - \dot{\lambda}_{\beta} \frac{\partial^3 f_{\beta}}{\partial \ddot{q}_s \partial t \partial \dot{q}_i} - \\ \lambda_{\beta} \frac{\partial^2 f_{\beta}}{\partial \ddot{q}_s \partial q_i} - \lambda_{\beta} \frac{\partial^3 f_{\beta}}{\partial \ddot{q}_s \partial \dot{q}_i \partial t}, (s, i = 1, 2, \dots, n) \quad (35) \end{aligned}$$

若令式(31)等于守恒量式(28), 可得

$$\begin{aligned} \xi_0^{\sigma} = \left\{ I^{\sigma} \cdot \exp \left(\int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) - \left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_{\beta} \frac{\partial f_{\beta}}{\partial \dot{q}_s} \right) \bar{\xi}_s^{\sigma} - \right. \\ \left. \lambda_{\beta} \frac{\partial f_{\beta}}{\partial \ddot{q}_s} (\dot{\xi}_s^{\sigma} - \dot{q}_s \dot{\xi}_0^{\sigma} - \ddot{q}_s \xi_0^{\sigma}) + G^{\sigma} - \right. \end{aligned}$$

$$\left. \frac{d}{dt} \left[\lambda_{\beta} \frac{\partial f_{\beta}}{\partial \ddot{q}_s} \exp \left(\int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] \bar{\xi}_s^{\sigma} \right\} (L + \lambda_{\beta} f_{\beta})^{-1} \quad (36)$$

因此, 在已知 $I^{\sigma}(t, q, \dot{q}, z) = \text{const.}$, 并选定合适的规范函数后, 可由式(34)和式(36)找到相应生成元函数 $\bar{\xi}_0^{\sigma}$ 和 $\bar{\xi}_s^{\sigma}$. 于是有如下定理:

定理 3 对于二阶非完整 Vacco 动力学的 Herglotz 型系统式(12), 若存在 r 个线性独立的第一积分式(31), 那么系统存在由方程式(34)和式(36)确定的生成元函数使其满足 Herglotz 型 Noether 准对称性.

若 $G^{\sigma} \equiv 0$, 则式(36)成为

$$\begin{aligned} \xi_0^{\sigma} = \left\{ I^{\sigma} \cdot \exp \left(\int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) - \left(\frac{\partial L}{\partial \dot{q}_s} + \lambda_{\beta} \frac{\partial f_{\beta}}{\partial \dot{q}_s} \right) \bar{\xi}_s^{\sigma} - \right. \\ \left. \lambda_{\beta} \frac{\partial f_{\beta}}{\partial \ddot{q}_s} (\dot{\xi}_s^{\sigma} - \dot{q}_s \dot{\xi}_0^{\sigma} - \ddot{q}_s \xi_0^{\sigma}) - \right. \\ \left. \frac{d}{dt} \left[\lambda_{\beta} \frac{\partial f_{\beta}}{\partial \ddot{q}_s} \exp \left(\int_{t_0}^t \frac{\partial L}{\partial z} d\theta \right) \right] \bar{\xi}_s^{\sigma} \right\} (L + \lambda_{\beta} f_{\beta})^{-1} \quad (37) \end{aligned}$$

此时, 定理 3 退化为:

定理 4 对于二阶非完整 Vacco 动力学的 Herglotz 型系统式(12), 若存在 r 个线性独立的第一积分式(31), 那么该系统存在由方程式(34)和式(37)确定的生成元函数使其满足 Herglotz 型 Noether 对称性.

定理 3 和定理 4 可称为二阶非完整 Vacco 动力学系统的 Herglotz 型 Noether 逆定理.

4 算例

一单位质量的质点在空间中运动. 若该质点受到非保守力为

$$F_x = -\dot{x}, F_y = -\dot{y}, F_z = -\dot{z} \quad (38)$$

令 $q_1 = x, q_2 = y, q_3 = z$, 它的运动受有一个双面理想二阶线性非完整约束为

$$f = -q_1 \ddot{q}_1 - q_2 \ddot{q}_2 + q_3 \ddot{q}_3 = 0 \quad (39)$$

试研究该系统的守恒量.

如果已知系统的 Herglotz 型 Lagrange 函数为

$$L = \frac{1}{2} (\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - z \quad (40)$$

将式(39)、式(40)均代入二阶非完整 Vacco 动力学的 Herglotz 型系统式(12)可得

$$e^{\int_{t_0}^t (-2\lambda \ddot{q}_1 - \dot{q}_1 + \dot{q}_1 - \ddot{\lambda} q_1 - 2\dot{\lambda} \dot{q}_1 + 2\dot{\lambda} q_1 +$$

$$2\lambda\dot{q}_1 - \lambda q_1 = 0 \quad (41)$$

$$e^{\iota-\iota_0} (-2\lambda\ddot{q}_2 - \ddot{q}_2 + \dot{q}_2 - \ddot{\lambda}q_2 - 2\dot{\lambda}\dot{q}_2 + 2\dot{\lambda}q_2 + 2\lambda\dot{q}_2 - \lambda q_2) = 0 \quad (42)$$

$$e^{\iota-\iota_0} (2\lambda\ddot{q}_3 - \ddot{q}_3 + \dot{q}_3 + \ddot{\lambda}q_3 + 2\dot{\lambda}\dot{q}_3 - 2\dot{\lambda}q_3 - 2\lambda\dot{q}_3 + \lambda q_3) = 0 \quad (43)$$

根据判据方程式(25)可得

$$\begin{aligned} & \lambda(-\ddot{q}_1\dot{\xi}_1 - \ddot{q}_2\dot{\xi}_2 + \ddot{q}_3\dot{\xi}_3) + \dot{\lambda}(-q_1\ddot{q}_1 - q_2\ddot{q}_2 + q_3\ddot{q}_3)\dot{\xi}_0 + \\ & \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 - 2z)\dot{\xi}_0 - \lambda(q_1\ddot{q}_1 + q_2\ddot{q}_2 - q_3\ddot{q}_3)\dot{\xi}_0 + \\ & \dot{q}_1(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) + \dot{q}_2(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0) + \dot{q}_3(\dot{\xi}_3 - \dot{q}_3\dot{\xi}_0) - \\ & \lambda q_1(\ddot{\xi}_1 - 2\ddot{q}_1\dot{\xi}_0 - \dot{q}_1\ddot{\xi}_0) - \lambda q_2(\ddot{\xi}_2 - 2\ddot{q}_2\dot{\xi}_0 - \dot{q}_2\ddot{\xi}_0) + \\ & \lambda q_3(\ddot{\xi}_3 - 2\ddot{q}_3\dot{\xi}_0 - \dot{q}_3\ddot{\xi}_0) - G - \dot{G} = 0 \end{aligned} \quad (44)$$

上述方程有解

$$\begin{aligned} & \xi_0 = 0, \xi_1 = 2\dot{q}_1, \xi_2 = 2\dot{q}_2, \xi_3 = 2\dot{q}_3, \\ & G = \dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 - 2z \end{aligned} \quad (45)$$

由定理 1, 将方程式(45)代入式(28), 可得相应的守恒量为

$$I_1 = e^{\iota-\iota_0} [(1+2\lambda)(\dot{q}_1^2 + \dot{q}_2^2) + (1-2\lambda)\dot{q}_3^2 + 2z + (2\dot{\lambda} - 2\lambda)(q_1\dot{q}_1 + q_2\dot{q}_2 - q_3\dot{q}_3)] = \text{const.} \quad (46)$$

特殊的, 由于 λ 的初值可以任意选取, 因而能

将式(46)中的 $\dot{\lambda}$ 换成 λ , 此时守恒量变成

$$I_1 = e^{\iota-\iota_0} [(1+2\lambda)(\dot{q}_1^2 + \dot{q}_2^2) + (1-2\lambda)\dot{q}_3^2 + 2z] = \text{const.} \quad (47)$$

其次, 当 $G \equiv 0$ 时, 由判据方程式(23)可得

$$\begin{aligned} & -\lambda(\ddot{q}_1\dot{\xi}_1 + \ddot{q}_2\dot{\xi}_2 - \ddot{q}_3\dot{\xi}_3) + \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 - \\ & 2z)\dot{\xi}_0 - \lambda(q_1\ddot{q}_1 + q_2\ddot{q}_2 - q_3\ddot{q}_3)\dot{\xi}_0 - \dot{\lambda}(q_1\ddot{q}_1 + \\ & q_2\ddot{q}_2 - q_3\ddot{q}_3)\dot{\xi}_0 + \dot{q}_1(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) + \dot{q}_2(\dot{\xi}_2 - \\ & \dot{q}_2\dot{\xi}_0) + \dot{q}_3(\dot{\xi}_3 - \dot{q}_3\dot{\xi}_0) - \lambda q_1(\ddot{\xi}_1 - 2\ddot{q}_1\dot{\xi}_0 - \\ & \dot{q}_1\ddot{\xi}_0) - \lambda q_2(\ddot{\xi}_2 - 2\ddot{q}_2\dot{\xi}_0 - \dot{q}_2\ddot{\xi}_0) + \lambda q_3(\ddot{\xi}_3 - \\ & 2\ddot{q}_3\dot{\xi}_0 - \dot{q}_3\ddot{\xi}_0) = 0 \end{aligned} \quad (48)$$

方程(48)有解

$$\xi_0 = 1, \xi_1 = 0, \xi_2 = 0, \xi_3 = 0 \quad (49)$$

由定理 2, 将方程式(49)的特解代入式(27), 可得相应守恒量为

$$\begin{aligned} I_2 = e^{\iota-\iota_0} & \left[\left(\frac{1}{2} + \lambda \right) \dot{q}_1^2 + \left(\frac{1}{2} + \lambda \right) \dot{q}_2^2 + \right. \\ & \left(\frac{1}{2} - \lambda \right) \dot{q}_3^2 + z - (\lambda - \dot{\lambda})(q_1\dot{q}_1 + q_2\dot{q}_2 + \\ & q_3\dot{q}_3) \left. \right] = \text{const.} \end{aligned} \quad (50)$$

最后, 根据已知守恒量由定理 3 寻找相应的无穷小生成元函数. 假设系统有守恒量如下

$$I = e^{\iota-\iota_0} [(1+2\lambda)(\dot{q}_1^2 + \dot{q}_2^2) + (1-2\lambda)\dot{q}_3^2 + 2z] = \text{const.} \quad (51)$$

再由式(35)可知

$$\begin{aligned} b_{11} &= 1 + \lambda, b_{22} = 1 + \lambda, b_{33} = 1 - \lambda, \\ \tilde{b}_{11} &= \frac{1}{1 + \lambda}, \tilde{b}_{22} = \frac{1}{1 + \lambda}, \tilde{b}_{33} = \frac{1}{1 - \lambda} \end{aligned} \quad (52)$$

将式(51)、式(52)代入式(34)分别可得

$$\bar{\xi}_1 = \tilde{b}_{11} \cdot \exp\left(\int_{\iota_0}^{\iota} \frac{\partial L}{\partial z} d\theta\right) \cdot \frac{\partial I}{\partial \dot{q}_1} = \frac{2+4\lambda}{1+\lambda} \dot{q}_1 \quad (53)$$

$$\bar{\xi}_2 = \tilde{b}_{22} \cdot \exp\left(\int_{\iota_0}^{\iota} \frac{\partial L}{\partial z} d\theta\right) \cdot \frac{\partial I}{\partial \dot{q}_2} = \frac{2+4\lambda}{1+\lambda} \dot{q}_2 \quad (54)$$

$$\bar{\xi}_3 = \tilde{b}_{33} \cdot \exp\left(\int_{\iota_0}^{\iota} \frac{\partial L}{\partial z} d\theta\right) \cdot \frac{\partial I}{\partial \dot{q}_3} = \frac{2-4\lambda}{1-\lambda} \dot{q}_3 \quad (55)$$

将式(51)~式(55)均代入式(36), 并选取

$$G = (1+2\lambda)(\dot{q}_1^2 + \dot{q}_2^2) + (1-2\lambda)\dot{q}_3^2 - 2z \quad (56)$$

则方程(53)~(56)有解

$$\begin{aligned} \xi_0 &= 0, \xi_1 = \frac{(2+4\lambda)\dot{q}_1}{1+\lambda} \\ \xi_2 &= \frac{(2+4\lambda)\dot{q}_2}{1+\lambda}, \xi_3 = \frac{(2-4\lambda)\dot{q}_3}{1-\lambda} \end{aligned} \quad (57)$$

5 结论

本文基于 Herglotz 变分原理研究了二阶非完整系统的 Vacco 动力学及其 Noether 定理, 主要结果如下: 首先, 基于 Herglotz 变分问题建立了二阶非完整系统 Vacco 动力学的 Herglotz 型运动微分方程式(12); 其次, 根据 Hamilton-Herglotz 作用量变分的两个基本公式(20)和式(21), 推导了二阶非完整系统 Vacco 动力学的 Herglotz 型 Noether 定理(定理 1 和定理 2); 最后, 还给出了二阶非完整系统 Vacco 动力学的 Herglotz 型 Noether 定理的逆定理(定理 3 和定理 4). 本文的研究方法和结果, 可以进一步推广至任意阶非完整约束系统.

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