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两端带有弹簧支撑的轴向运动梁振动分析*

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摘要 研究外部激励下,两端带有弹簧支撑的轴向运动梁的横向振动.运用广义哈密顿原理,推导得到运动梁的控制方程.通过数值方法研究系统的固有频率和模态,解析分析得到轴向运动梁的临界速度表达式并考虑弹簧刚度对临界速度的影响.发现临界速度随弹簧刚度增大而收敛于某一值.运用 Galerkin 截断法数值研究两端带有弹簧支撑的轴向运动梁的稳态幅频响应曲线,并通过计算力传递率来研究系统的隔振效果.发现力传递率在高频外激励下存在多个峰值.

关键词 轴向运动梁, 弹性支撑, 临界速度, 幅频响应, 传递率

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引言

工业生产和生活中轴向运动梁结构随处可见,例如机械传送带、空中缆车索道、发动机传送带等传动系统.当系统的弯曲刚度不可忽略时,通常可以简化成轴向运动梁模型.因此其振动分析及控制有着重要的研究意义.轴向运动 Euler 梁是最普通的轴向运动连续体模型,这种模型不考虑梁的剪切应力和截面扭转,因此梁的控制方程易于求解和分析.自 20 世纪 60 年代 Mote 院士提出动力学模型以来[1,2],国内外学者对轴向运动弦线、Euler 梁、Timoshenko 梁在不同边界条件下的建模、横向振动分析、稳定性分析等进行了广泛的研究[3-7].

在工程中,振动往往是普遍存在的、有害的.由阻尼和弹簧组成的被动隔振装置因为其简单、无附加能量和经济实用的优势,成为了解决这一问题的首选方案.2008 年 Ibrahim 对非线性隔振作出了研究[8].Lang 的研究结果表明立方非线性的粘性阻尼对单自由度系统有较好的隔振效果[9].Laalej 通过实验[10]、Peng 通过谐波平衡法[11]都验证了非线性阻尼对单自由度系统有较好的隔振效果.

通过以上文献的研究发现,被动隔振装置的研究受到广泛的关注.在各种激励作用下,弹性梁的弯曲振动往往表现出明显的非线性特征[12,13].然而,隔振研究很少考虑到结构的振动例如弯曲振动.作为工程中基本的结构单元之一,对于考虑弹性梁结构的横向弯曲振动和边界条件的隔振研究很有必要[14].

杨晓东等人研究了带有扭转弹簧两端铰支的轴向运动梁[15].Ding 等对于静态轴向连续体引入了竖直弹簧支撑以研究隔振效果[16,17].

本文通过对两端弹簧支撑的轴向运动梁在亚临界状态下的受迫振动分析,研究其振动特性和隔振效果.

1 建立力学模型

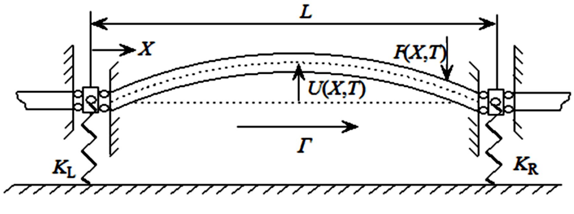

图 1 轴向运动梁的物理模型

Fig.1 The physical model of an axially moving beam

图 1 所示为两端带有弹簧支撑的轴向运动梁的物理模型.其中轴向坐标 X 是与左边界的距离,T 是时间坐标,$U(X,T)$ 是横向位移,$F(X,T)$ 是外部激励,K_L 和 K_R 分别为左右弹簧的刚度,此外,梁以

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固定速度 Γ 作轴向运动,梁长为 L,梁的初始轴向张力为 P.

系统总应变能 V_p、动能 T_k 和外力做的虚功 δW 分别为:

$$V_p=\frac{1}{2}\int_0^L\left[PU,_X^2+\frac{1}{2}EAU,_X^4+EIU,_{XX}^2\right]\mathrm{d}X+\frac{1}{2}\left[K_L[U(0)]^2+K_R[U(L)]^2\right] \tag{1}$$

$$T_k=\frac{1}{2}\int_0^L\rho AU,_T^2\mathrm{d}X=\frac{1}{2}\int_0^L\rho A(U,_T+\Gamma U,_X)^2\mathrm{d}X \tag{2}$$

$$\delta W=\int_0^L F\cos(\Omega t)\delta U\mathrm{d}X \tag{3}$$

根据哈密顿原理 $\delta\int_{t_1}^{t_2}(T_k-V_p)\mathrm{d}T+\int_{t_1}^{t_2}\delta W\mathrm{d}T=0$ 得到运动控制方程,如下:

$$\rho A(U,_{TT}+2\Gamma U,_{XT}+\Gamma,_T U,_X+\Gamma^2U,_{XX})-PU,_{XX}+EIU,_{XXXX}-\frac{3}{2}EAU,_X^2U,_{XX}=F\cos(\Omega t) \tag{4}$$

其中,轴力不变 $P=P_0$,轴向速度不变 $\Gamma,_T=0$,边界条件为:

$$U,_X(0,T)=0,U,_X(L,T)=0,$$
$$EIU,_{XXX}(0,T)+K_LU(0,T)=0,$$
$$EIU,_{XXX}(L,T)-K_RU(L,T)=0 \tag{5}$$

引入无量纲参数:

$$u=\frac{U}{L},\ x=\frac{X}{L},\ t=T\sqrt{\frac{P_0}{\rho AL^2}},$$
$$\gamma=\Gamma\sqrt{\frac{\rho A}{P_0}},\ k_f^2=\frac{EI}{P_0L^2},\ k_1=\frac{EA}{P_0},$$
$$\mu=c\sqrt{\frac{L^2}{P_0\rho A}},\ \omega_b=\Omega\sqrt{\frac{\rho AL^2}{P_0}}, \tag{6}$$
$$f=F\frac{1}{P_0},\ k_L=\frac{K_LL^3}{EI},\ k_R=\frac{K_RL^3}{EI}$$

将无量纲化参数(6)代入控制方程(4)得到:

$$u,_{tt}+2\gamma u,_{xt}+(\gamma^2-1)u,_{xx}+k_f^2u,_{xxxx}+\mu u,_t-\frac{3}{2}k_1u,_x^2u,_{xx}=f\cos(\omega_bt) \tag{7}$$

方程(7)的线性派生系统为:

$$u,_{tt}+2\gamma u,_{xt}+(\gamma^2-1)u,_{xx}+k_f^2u,_{xxxx}=0 \tag{8}$$

无量纲化后的边界条件为:

$$u,_x(0)=0,\ u,_x(1)=0,$$
$$u,_{xxx}(0)+k_Lu(0)=0,\ u,_{xxx}(1)-k_Ru(1)=0 \tag{9}$$

2 固有频率

假设(8)的解为 $u(x,t)=\phi_n(x)e^{i\omega_kt}$,代入式(8)和式(9),得到:

$$\omega_n^2\phi_n-2i\gamma\omega_n\phi_n'-(\gamma^2-1)\phi_n''-k_f^2\phi_n^{(4)}=0 \tag{10}$$

$$\phi_n'(0)=0,\ \phi_n'(1)=0,$$
$$\phi_n'''(0)+k_L\phi_n(0)=0,\ \phi_n'''(1)-k_R\phi_n(1)=0 \tag{11}$$

式(10)的特征方程为:

$$k_f^2\beta^4+(1-\gamma^2)\beta^2-2\gamma\omega_n\beta-\omega_n^2=0 \tag{12}$$

式(12)的假设解为:

$$\phi_n(x)=C_{1n}(e^{i\beta_{1n}x}+C_{2n}e^{i\beta_{2n}x}+C_{3n}e^{i\beta_{3n}x}+C_{4n}e^{i\beta_{4n}x}) \tag{13}$$

将式(13)代入(11)式,得到:

$$\begin{pmatrix}\beta_{1n} & \cdots & \cdots & \beta_{4n}\\ \beta_{1n}e^{i\beta_{1n}} & \cdots & \cdots & \beta_{4n}e^{i\beta_{4n}}\\ -i\beta_{1n}^3+k_L & \cdots & \cdots & -i\beta_{4n}^3+k_L\\ -i\beta_{1n}^3e^{i\beta_{1n}}-k_Re^{i\beta_{1n}} & \cdots & \cdots & -i\beta_{4n}^3e^{i\beta_{4n}}-k_Re^{i\beta_{4n}}\end{pmatrix}\cdot\begin{pmatrix}1\\ C_{2n}\\ C_{3n}\\ C_{4n}\end{pmatrix}C_{1n}=0 \tag{14}$$

系统的物理参数如表1所示。

表1 轴向运动梁的物理参数

Table 1 Physical parameters of an axially moving beam

Name	Symbol	Value
Young's modulus	E	68.9 GPa
Density	ρ	2800 kg/m^3
Length	L	0.5 m
Initial force	P_0	100 N
Area of cross-section	A	10^{-4} m^2
Inertial moment	I	2.083×10^{-10} m^4
Damping coefficient	C	10 N s/m^2
Stiffness of spring	K_L & K_R	5741.67 N/m

要使方程(14)有非零解,其系数行列式必须有零解,从而解出系统的固有频率.图 2 和图 3 显示了弹簧刚度和轴向速度对前两阶固有频率的影响.

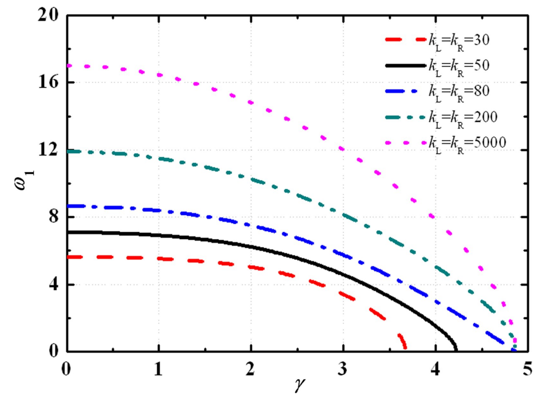

图 2　第一阶固有频率

Fig.2　The first natural frequency

图 2 表示,一阶固有频率随轴向速度增大而变小直至为零,随着弹簧刚度增大,临界速度最终收敛于某一值,即弹簧刚度无限大,相当于两端固支边界条件时的临界速度.

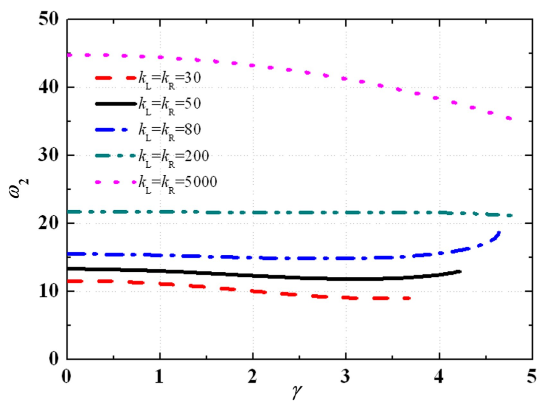

图 3　第二阶固有频率

Fig.3　The second natural frequency

图 3 表示,在亚临界状态下,第二阶固有频率在两端弹簧刚度很小的时候随着轴向速度先减小后增大,当弹簧刚度足够大,频率随着轴向速度增加而减小.

3　临界速度

考虑临界速度时,由于位移 u 不依赖于时间,因此将方程(8)中与时间有关的项忽略,得到方程:

$$(\gamma^2-1)u_{,xx}+k_f^2u_{,xxxx}=0 \tag{15}$$

式(15)的特征方程为 $(\gamma^2-1)\lambda^2+k_f^2\lambda^4=0$,当 $\gamma^2>1$ 时,特征方程的四个根为 $\lambda_{1,2}=0,\lambda_{3,4}=\pm\alpha\mathrm{i}$,其中 $\alpha=\sqrt{\dfrac{\gamma^2-1}{k_f^2}}$,则式(15)的解为:

$$u(x)=C_1+C_2x+C_3\cos\alpha x+C_4\sin\alpha x \tag{16}$$

将式(16)代入边界条件(9),有:

$$\begin{pmatrix} 0 & 1 & 0 & \alpha \\ 0 & 1 & -\alpha\sin\alpha & \alpha\cos\alpha \\ k_L & 0 & k_L & -\alpha^3 \\ k_R & k_R & k_R\cos\alpha-\alpha^3\sin\alpha & k_R\sin\alpha+\alpha^3\cos\alpha \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}=0 \tag{17}$$

考虑两边弹簧刚度相同,记 $k_L=k_R=p$.

要使方程(17)有解,其系数行列式必须有零解,则:

$$\cos\alpha=\frac{4p^2-p^2\alpha^2+4p\alpha^4-4\alpha^6}{4p^2+p^2\alpha^2-4p\alpha^4+4\alpha^6} \tag{18}$$

记方程(18)的一组解为 $0,\alpha_1,\alpha_2,\alpha_3\cdots$.

因为其中 α_1 与第一阶临界速度 γ_{cr} 满足 $\alpha_1=\sqrt{\dfrac{\gamma_{cr}^2-1}{k_f^2}}$,所以推导得出:

$$\gamma_{cr}=\sqrt{k_f^2\alpha_1^2+1} \tag{19}$$

由式(18)和式(19)可以求解得到临界速度与弹簧刚度的关系.如图 4 所示,随着弹簧刚度增加,临界速度增大,当弹簧刚度增大至 $k_L=k_R=80$ 时,临界速度不再随之增大.

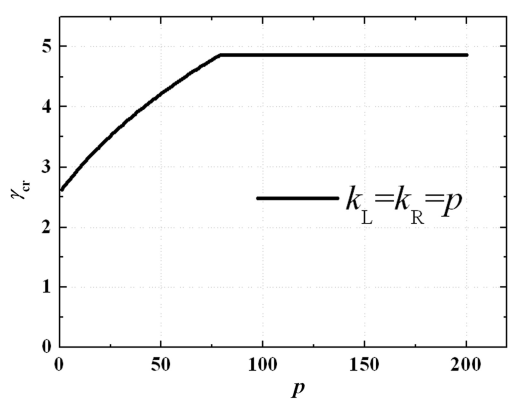

图 4　临界速度与弹簧刚度的关系

Fig.4　Effects of spring stiffness on critical velocity

4　稳态响应

为求得运动系统的稳态响应,运用 Galerkin 截断法对静态梁进行离散化处理.静态梁的线性派生

系统[16]为:

$$u_{,tt}+k_f^2 u_{,xxxx}=0 \tag{20}$$

假设式(20)的解为 $u(x,t)=\phi(x)\sin(\omega t+\theta)$,其特征方程为 $k_f^2\beta^4-\omega^2=0$,特征根为 $\beta^4=\dfrac{\omega^2}{k_f^2}$,所以模态函数为:

$$\phi(x)=D_1\cos\beta x+D_2\sin\beta x+D_3\mathrm{ch}\beta x+D_4\mathrm{sh}\beta x \tag{21}$$

将式(21)代入式(11)可以求得系数 D_1,D_2,D_3,D_4.运用数值计算,画出稳态的幅频响应,图中 $A_1(x)$ 表示 x 位置的稳态幅值,ω_b 表示外激励频率.

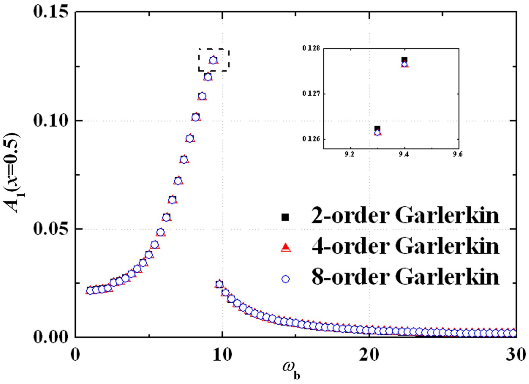

图 5 2,4,8 阶截断的稳态响应

Fig.5 The steady-state response of 2,4,8 truncations

图 5 描述了 2 阶、4 阶和 8 阶 Galerkin 截断得到的系统中点 $x=0.5$ 处的稳态幅频响应曲线.观察发现当 $\omega_b=9.4$ 时,2 阶 Galerkin 截断的幅值为 $A_1=0.127755$.4 阶截断和 8 阶截断的幅值相等,都为 $A_1=0.12765$,误差 $\eta=0.082\%$,所以 4 阶 Galerkin 截断足够精确.

5 力的传递率

当系统进入稳态后,在外力作用下,瞬时的力传递率可以定义为:

$$\eta_f(t)=\frac{F_f(t)}{F_0}=\frac{EI}{F_0L^2}[k_L u(0,t)+k_R u(1,t)] \tag{22}$$

由此可以定义系统的传递率为:

$$\eta_{fs}=\mathrm{Max}[\eta_f(t)] \tag{23}$$

令右边弹簧的刚度 $k_R=50$ 不变,改变左边弹簧刚度 k_L 的值,作正反扫频,结果如图 6、图 7 和图 8 所示.图中空心符号代表正向扫频,实心符号代表反向扫频.

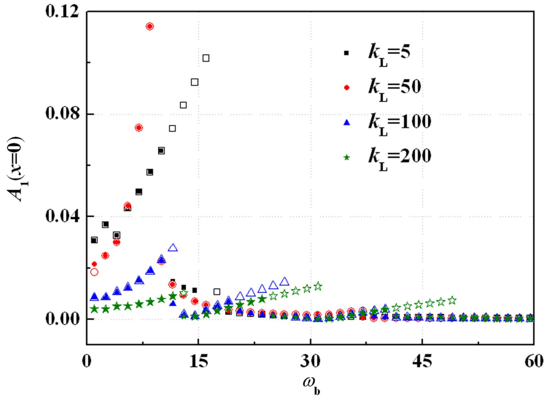

图 6 幅频响应($x=0$)

Fig.6 The amplitude-frequency response($x=0$)

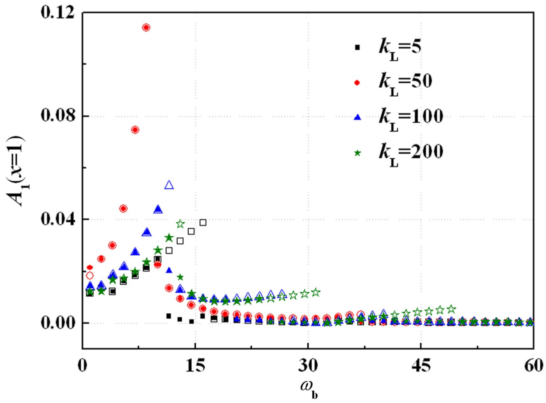

图 7 幅频响应($x=1$)

Fig.7 The amplitude-frequency response($x=1$)

图 6 和图 7 描述了右端弹簧刚度 $k_R=50$ 时,轴向运动梁左端($x=0$)和右端($x=1$)的稳态振动幅值受左端弹簧刚度 k_L 的影响.从中可以发现,在两端弹簧刚度相等时两端的幅值最大.

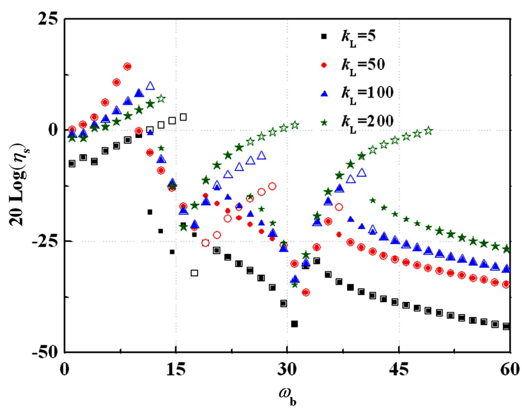

图 8 力传递率

Fig.8 Force transmissibility

观察图 8 发现,系统传递率随着外激励频率的增大出现多个峰值.对比图 6、7 和图 8,发现仅通过

稳态响应并不能判断高频外激励下的隔振效果.而引入力传递率后,系统在高频外激励下的隔振效果更直观.

6　小结

本文通过数值方法分析了两端带有弹簧支撑的轴向运动梁的受迫振动,主要研究了系统的固有频率、模态、临界速度、稳态响应和力传递率,得到了临界速度的解析式.发现在一定范围内,临界速度随弹簧刚度增大而增大.发现系统的力传递率在高频外激励下存在多个峰值,所以系统的隔振效果通过力传递率来分析更为准确.

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NONLINEAR TRANSVERSE VIBRATION OF AN AXIALLY MOVING BEAM WITH VERTICAL SPRING BOUNDARY *

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Abstract The transverse vibration of an axially moving elastic beam supported by vertical-spring boundary under external excitation was studied. By using the generalized Hamilton principle, the governing equation of motion was obtained. Moreover, the modal analysis was conducted by using analytic method, and the relationship between natural frequency and axial velocity was revealed. The expression of the critical velocity of the axially moving beam was derived, and the effect of the spring stiffness on the critical velocity was studied. The critical velocity converges to a certain value with the increase of the spring stiffness. Furthermore, the amplitude-frequency responses in steady state were obtained by using the Galerkin truncation method. The vibration isolation performances of the system were evaluated by the force transmissibility. There are multiple peaks of the force transmissibility under high-frequency external excitations.

Key words axially moving beam, elastic support, critical velocity, amplitude frequency response, transmissibility

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