混合非完整系统问题*

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摘要 研究一类混合非完整系统的运动.它可分为3个阶段:第1阶段为完整系统的连续运动,第2阶段为冲击运动,第3阶段为非完整系统的连续运动.后一阶段的初始条件由前一阶段的运动终了条件确定.举例说明结果的应用.

关键词 完整系统,非完整系统,冲击运动,混合非完整系统

引言

一般说来,完整系统的运动和非完整系统的运动,都是各自独立进行研究的.对系统的冲击运动,也是分为完整系统的冲击运动和非完整系统的冲击运动,如文献[1,2]研究了完整系统的冲击运动,文献[3~5]研究了完整系统和非完整系统的冲击运动.实际中存在一类系统,称为混合非完整系统[6].它是指一个完整系统的运动在某时刻突然施加非完整约束.文献[6]用几何方法,主要是余分布,研究了混合非完整系统的运动.本文用简单分析方法研究这类系统的运动.这类系统的运动可分为3个阶段.第1阶段是完整系统的连续运动;第2阶段是冲击运动,在某时刻突加非完整约束,使广义速度发生跃变;第3阶段为非完整系统的连续运动,以冲击后的结果为初始条件,因非完整约束的加入而使自由度减少.

1 完整系统的连续运动

一般完整系统的运动微分方程有形式

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s \quad (s=1,\cdots,n) \tag{1}$$

其中 $L=L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 为系统的 Lagrange 函数,$Q_s=Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 为非势广义力.假设运动的初始条件为

$$t=t_0, q_s=q_{s0}, \dot{q}_s=\dot{q}_{s0} \tag{2}$$

则方程(1)的解可表为

$$q_s=q_s(t,t_0,q_{k0},\dot{q}_{k0})$$

$$\dot{q}_s=\dot{q}_s(t,t_0,q_{k0},\dot{q}_{k0}) \quad (s,k=1,\cdots,n) \tag{3}$$

这个运动一直延续到 $t=\bar{t}$,此时有

$$q_{s1}=q_s(\bar{t},t_0,q_{k0},\dot{q}_{k0})$$

$$\dot{q}_{s1}=q_s(\bar{t},t_0,q_{k0},\dot{q}_{k0}) \tag{4}$$

2 冲击运动

假设在 $t=\bar{t}$ 时,完整系统(1)突然施加非完整约束

$$f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})=0 \quad (\beta=1,\cdots,g) \tag{5}$$

方程(1)的冲击运动方程表为

$$\left(\frac{\partial L}{\partial \dot{q}_s}\right)_2-\left(\frac{\partial L}{\partial \dot{q}_s}\right)_1=P_s \quad (s=1,\cdots,n) \tag{6}$$

其中 P_s 为广义冲击力的冲量,下杯1表示冲击前的值,下标2表示冲击后的值.假设冲击后非完整约束(5)仍保持,则有

$$f_\beta=(\bar{t},\boldsymbol{q}_2,\dot{\boldsymbol{q}}_2)=0 \tag{7}$$

由冲击前的条件(4)和代数方程(6),(7)可求得冲击后的广义速度 $\dot{q}_{s2}$.注意到,冲击过程中坐标是不改变的,即有

$$q_{s2}=q_{s1} \tag{8}$$

3 非完整系统的连续运动

冲击后的连续运动方程表为

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s} \tag{9}$$

其中 λ_β 为约束乘子,而初始条件为

$$t=\bar{t}, q_s=q_{s2}, \dot{q}_s=\dot{q}_{s2} \tag{10}$$

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由方程(9),(5)在初始条件(10)下求解,可最终求得

$$q_s = q_s(t, \bar{t}, t_0, q_{k0}, \dot{q}_{k0})$$
$$\dot{q}_s = \dot{q}_s(t, \bar{t}, t_0, q_{k0}, \dot{q}_{k0}) \tag{11}$$

4 算例

一半径为 R 的匀质圆球在一水平面上自由运动.该平面分成两部分,一部分是光滑的,另一部分是粗糙的.试研究该混合非完整系统的运动.

首先,研究完整系统的连续运动,即球在光滑平面部分的运动.取球心水平坐标 x, y 和 Euler 角 ψ, θ, φ 为广义坐标.假设平面在 $x < 0$ 时是光滑的,而在 $x > 0$ 时是绝对粗糙的.系统的 Lagrange 函数为

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}mk^2(\dot{\psi}^2 + \dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta) \tag{12}$$

其中 m 为球的质量,k 为球的回转半径.取球的角速度在与固定坐标系 $Oxyz$ 相平行的轴上的投影 $\omega_1, \omega_2, \omega_3$ 以及 $\dot{x}, \dot{y}$ 为准速度,则有

$$\omega_1 = \dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi$$
$$\omega_2 = \dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi$$
$$\omega_3 = \dot{\psi} + \dot{\varphi}\cos\theta$$
$$\omega_4 = \dot{x}$$
$$\omega_5 = \dot{y} \tag{13}$$

准速度表示的 Lagrange 函数为

$$L^* = \frac{1}{2}m(\omega_4^2 + \omega_5^2) + \frac{1}{2}mk^2(\omega_1^2 + \omega_2^2 + \omega_3^2) \tag{14}$$

方程(1)在准坐标、准速度下表为

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \omega_s} - \frac{\partial L^*}{\partial \pi_s} + \frac{\partial L^*}{\partial \omega_k}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \omega_k}{\partial \dot{q}_r} - \frac{\partial \omega_k}{\partial q_r}\right)\frac{\partial \dot{q}_r}{\partial \omega_s} = P_s^*, (s = 1, \cdots, 5) \tag{15}$$

其中 $\omega_k = \omega_k(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ 为准速度,π_s 为准坐标.通过计算,得

$$\frac{\partial L^*}{\partial \omega_k}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \omega_k}{\partial \dot{q}_r} - \frac{\partial \omega_k}{\partial q_r}\right)\frac{\partial \dot{q}_r}{\partial \omega_s} = 0$$
$$(s, k, r = 1, \cdots, 5)$$
$$\frac{\partial L^*}{\partial \pi_s} = 0, P_s^* = 0 \quad (s = 1, \cdots, 5)$$
$$\frac{\partial L^*}{\partial \omega_s} = mk^2\omega_s \quad (s = 1, 2, 3),$$
$$\frac{\partial L^*}{\partial \omega_k} = m\omega_k \quad (k = 4, 5)$$

于是有

$$m\ddot{x} = 0, m\ddot{y} = 0, mk^2\dot{\omega}_1 = 0,$$
$$mk^2\dot{\omega}_2 = 0, mk^2\dot{\omega}_3 = 0 \tag{16}$$

这是完整系统连续运动的微分方程.假设运动的初始条件为[6]

$$t = 0, x = x_0 < 0, y = y_0,$$
$$\dot{x} = \dot{x}_0 > 0, \dot{y} = \dot{y}_0,$$
$$\omega_1 = \omega_{10}, \omega_2 = \omega_{20} > 0, \omega_3 = \omega_{30} \tag{17}$$

积分方程(16),得

$$x = \dot{x}_0 t + x_0, y = \dot{y}_0 t + y_0,$$
$$\omega_1 = \omega_{10}, \omega_2 = \omega_{20}, \omega_3 = \omega_{30} \tag{18}$$

当

$$\bar{t} = -\frac{x_0}{\dot{x}_0} \tag{19}$$

时,$x = 0$,即达到平面光滑部分与粗糙部分的分界线.

其次,研究冲击运动.与方程(15)相应的冲击运动方程给出

$$m[(\dot{x})_2 - (\dot{x})_1] = P_x,$$
$$m[(\dot{y})_2 - (\dot{y})_1] = P_y$$
$$mk^2[(\omega_1)_2 - (\omega_1)_1] = P_y R,$$
$$mk^2[(\omega_2)_2 - (\omega_2)_1] = -P_x R,$$
$$mk^2[(\omega_3)_2 - (\omega_3)_1] = 0 \tag{20}$$

其中 P_x, P_y 分别为 x, y 方向的冲击力的冲量,它们是由平面的粗糙性引起的摩擦力冲量.粗糙性表示的非完整约束是指球与平面接触点的速度为零,即

$$\dot{x} - \omega_2 R = 0, \dot{y} + \omega_1 R = 0 \tag{21}$$

假设冲击后,它们仍满足,即

$$(\dot{x})_2 - (\omega_2)_2 R = 0, (\dot{y})_2 + (\omega_1)_2 R = 0 \tag{22}$$

注意到

$$(\dot{x})_1 = \dot{x}_0, (\dot{y})_1 = \dot{y}_0,$$
$$(\omega_1)_1 = \omega_{10}, (\omega_2)_1 = \omega_{20}, (\omega_3)_1 = \omega_{30}$$

则由方程(20),(22)可解得

$$(\dot{x})_2 = \frac{R^2\dot{x}_0 + Rk^2\omega_{20}}{R^2 + k^2},$$
$$(\dot{y})_2 = \frac{R^2\dot{y}_0 + Rk^2\omega_{10}}{R^2 + k^2},$$
$$(\omega_1)_2 = \frac{-R^2\dot{y}_0 + k^2\omega_{10}}{R^2 + k^2},$$

$$(\omega_2)_2 = \frac{R^2\dot{x}_0 + k^2\omega_{20}}{R^2 + k^2},$$

$$(\omega_3)_2 = \omega_{30} \tag{23}$$

最后,研究非完整系统的连续运动.微分方程为

$$m\ddot{x} = \lambda_1, m\ddot{y} = \lambda_2, mk^2\dot{\omega}_1 = \lambda_2 R,$$

$$mk^2\dot{\omega}_2 = -\lambda_1 R, mk^2\dot{\omega}_3 = 0 \tag{24}$$

其中 λ_1, λ_2 为约束乘子.运动的初始条件为

$$t = \bar{t}, x = 0, y = \dot{y}_0\bar{t} + y_0,$$

$$\dot{x} = (\dot{x})_2, \dot{y} = (\dot{y})_2,$$

$$\omega_1 = (\omega_1)_2, \omega_2 = (\omega_2)_2, \omega_3 = (\omega_3)_2 \tag{25}$$

由式(21),(24)可解得

$$\lambda_1 = \lambda_2 = 0 \tag{26}$$

这就是文献[7,8]研究的非完整系统的自由运动.注意到式(26),(25),积分方程(24),得

$$x = \frac{R^2\dot{x}_0 + Rk^2\omega_{20}}{R^2 + k^2}(t - \bar{t}),$$

$$y = \frac{R^2\dot{y}_0 + Rk^2\omega_{10}}{R^2 + k^2}(t - \bar{t}) + \dot{y}_0\bar{t} + y_0,$$

$$\omega_1 = \frac{-R^2\dot{y}_0 + k^2\omega_{10}}{R^2 + k^2},$$

$$\omega_2 = \frac{R^2\dot{x}_0 + k^2\omega_{20}}{R^2 + k^2},$$

$$\omega_3 = \omega_{30} \tag{27}$$

这样,这个混合非完整系统的运动,在 $0 \leqslant t \leqslant \bar{t}$ 时由式(18)给出,而在 $\bar{t} \leqslant t < \infty$ 时由式(27)给出.所得结果与文献[6]用余分布的几何方法所得是一致的.文献[6]的结果从微分几何上来说是清楚的,而这里从力学上来说是清楚的.

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ON PROBLEM OF HYDRIC NONHOLONOMIC SYSTEMS*

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Abstract The intent of this paper is to study the motion of a kind of hydric nonholonomic systems. The motions of the system can be divided into three stages. The first stage is the continuous motion of a holonomic system. The second stage is an impulse motion. The third stage is the continuous motion of a nonholonomic system. The initial conditions of the last stage are detemined by the final condition of the preceding stage. An example is given to illustrate the application of the result.

Key words holonomic system, nonholonomic system, impulse motion, hydric nonholonomic system

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